Assignment 18

1. Find an example of a connected set which is not pathwise connected.

2. Assume that $x_0 \in \mathbb{R}^n$ and that $f \in \mathrm{C}^{1-}(\mathbb{R}^n, \mathbb{R}^n)$ is such that
$$|f(x) - f(y)| \leq L|x - y|\,,\ x, y \in \mathbb{R}^n$$
for some $L < 1$. Show that the operator
$$T : \mathrm{C}([0,1], \mathbb{R}^n) \to \mathrm{C}([0,1], \mathbb{R}^n)\,, u \to Tu$$
defined through
$$(Tu)(t) = x_0 + \int_0^t f\big(u(\tau)\big)\, d\tau\,,\ t \in [0,1]$$
has a fixed point in $\mathrm{C}([0,1], \mathbb{R}^n)$ which solves
$$\begin{cases} u' = f(u) \\ u(0) = x_0 \end{cases}.$$

3. Let $f \in \mathrm{C}(D, \mathbb{R}^m)$ for some $D \overset{o}{\subset} \mathbb{R}^n$ and assume it is differentiable at $x \in D$. Show that there exists a constant $M > 0$ such that
$$|f(x) - f(y)| \leq M\,|x - y|$$
for y in a neighborhood of x.

4. Let $f \in \mathrm{C}^1(D, \mathbb{R})$ for some $D \overset{o}{\subset} \mathbb{R}^n$. Fix $x \in D$ and assume that $\nabla f(x) \neq 0$. Show that $\nabla f(x)$ points in the direction of maximal growth of f.

5. You ask a question.

The Homework is due Friday, April 18.