Assignment 7

1. Let $f : D \to \mathbb{R}$ be differentiable at $x_0 \in D \overset{o}{\subset} \mathbb{R}$. Show that f is Lipschitz continuous at x_0.

2. Analyze the continuity and differentiability of the function
$$f(x) := \begin{cases} 0\,, & x \leq 0\,, \\ \frac{1}{x} e^{-\frac{1}{x^2}}\,, & x > 0\,. \end{cases}$$
What can you say about higher order derivatives at $x = 0$?

3. Let $f : D \to \mathbb{R}$ be differentiable at $x_0 \in D \overset{o}{\subset} \mathbb{R}$. Prove or disprove:
(i) f is strictly increasing at x_0 implies that $f'(x_0) > 0$.
(ii) $f'(x_0) = 0$ implies that f has a local minimum or maximum at x_0.
(iii)f is Lipschitz continuous at x_0 and has local minimum there implies that f is differentiable at x_0.

4. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$. Define
$$d^+ f(x_0) = \lim_{x \to x_0 + 0} \frac{f(x) - f(x_0)}{x - x_0}\,,\ x_0 \in [a, b)\,,$$
$$d^- f(x_0) = \lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0}\,,\ x_0 \in (a, b]\,,$$
the right and left derivative of f at x_0, respectively, if the limit exists.
(i) Show that f is differentiable at $x_0 \in (a, b)$ if and only if $d^{\pm} f(x_0)$ exist and coincide.
(ii) Assume that f takes on a minimum or a maximum at a or b and is one-side differentiable there. What can you say about $d^+ f(a)$ and $d^- f(b)$?

The Homework is due Monday, Novermber 25