Final Examination

Print your name: ________________ ________________

Print your ID #: ________________________________

You have 2 hours to solve the problems. Good luck!

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ and assume that $f(x) = O(|x - x_0|)$ and $g(x) = o(1)$ as $x \to x_0$. What can you say about the product fg as $x \to x_0$ and why?

2. Let $f : \mathbb{R} \to \mathbb{R}$ and consider

(A) f is differentiable at $x_0 \in \mathbb{R}$ with derivative $f'(x_0) \in \mathbb{R}$,
(B) $\lim_{h\to 0} \frac{1}{2h}\big[(f(x+h) - f(x-h)\big] = f'(x_0)$.

Does (A) imply (B)? Does (B) imply (A)? If your answer is yes, justify it; if it is no, give a counterexample.

3. Let $B \subset \mathbb{R}$ be bounded and $f : B \to \mathbb{R}$ be uniformly continuous. Show that f is bounded.

4. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences of reals. Denote the set containing their accumulation points by S_1 and S_2, respectively.

(a) Let a new sequence $(z_n)_{n\in\mathbb{N}}$ be given by $z_n = x_n + y_n$, $n \in \mathbb{N}$, and denote by S_3 the set of its accumulation points. Is $S_3 \subset S_1 + S_2$? Or $S_1 + S_2 \subset S_3$? Motivate your answers.

(b) Consider now the sequence $(x_1, x_2, y_1, x_3, x_4, y_2, x_5, y_6, \dots)$. Characterize the set of its accumulation points in terms of S_1 and S_2. Justify your answer.

5. Compute $\lim_{n\to\infty}\big[(27n^3+1)^{1/3} - 3n\big]$ and justify your answer.

6. Let $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ be such that

$\lim\limits_{|x| \to \infty} |f(x) - x| = 0$ that is, such that

$$\forall \varepsilon > 0 \; \exists \, R > 0 \text{ with } |f(x) - x| \le \varepsilon \; \forall \, x \in \mathbb{R} \text{ s.t. } |x| \ge R \, .$$

Prove that f is uniformly continuous.

7. Compute an approximation of $\sqrt{404}$ to one digit after the comma and give an estimate for the error incurred.

8. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and assume that $\lim_{x\to\infty} f(x) = 0$. Does it follow that $\lim_{x\to\infty} f'(x) = 0$? Justify your answer by giving a proof or by constructing a counterexample.

9. Let $f \in \mathrm{C}^1(\mathbb{R}, \mathbb{R})$ and $x_0, a, b \in \mathbb{R}$. Show that

$$|f(x) - a - b\,(x - x_0)| = o(|x - x_0|) \text{ as } x \to x_0$$

already implies that $a = f(x_0)$ and $b = f'(x_0)$.