Print your name: ______________________

Print your ID #: _______________________

You have 2 hours to solve the problems. Good luck!
1. Let $g \in C^1([c,d],[a,b])$ and $f \in C([a,b])$. Define

$$F(x) := \int_a^{g(x)} f(y) \, dy, \ x \in (a,b).$$

Show that $F$ is differentiable and compute its derivative.
2. Let \((x_n)_{n \in \mathbb{N}}\) be a decreasing sequence in \([0, \infty)\) and prove that
\[
\sum x_n < \infty \iff \sum 2^k x_{2^k} < \infty.
\]
3. Let a sequence of functions \((f_n)_{n \in \mathbb{N}}\) be defined by

\[f_n(x) = \cos(x)^n, \ x \in [0, \frac{\pi}{2}], \ n \in \mathbb{N}.
\]

Let \(g \in C([0, \frac{\pi}{2}])\) be such that \(g(0) = 0\). What is the limit of \((g f_n)_{n \in \mathbb{N}}\)?

Is the convergence pointwise? Is it uniform? Justify your answer.
4. Show that $f$ defined through

$$f(x) = \log^2(1 + x)$$

is analytic in a neighborhood of the origin. Compute the coefficients of its power series expansion about $x = 0$. 

5. Let $(M, d)$ be a metric space. For a subset $A \subset M$ define
\[ \bar{A} = A \cup LP(A) \]
and show that
\[ \bar{A} = \bigcap \{ B \subset M \mid A \subset B \text{ and } B \text{ is closed} \}. \]
6. Let $(M, d)$ be a metric space. For a subset $A \subset M$ define

$$\hat{A} := \{ x \in A | \exists r > 0 \text{ s.t. } B(x, r) \subset A \}.$$ 

Prove or disprove: $(A \cup B)^\circ = \hat{A} \cup \hat{B}$, $(A \cap B)^\circ = \hat{A} \cap \hat{B}$
7. Prove or disprove:

\[ \left\{ \frac{1}{\sqrt{n}} \tanh(nx) : \mathbb{R} \to \mathbb{R} \mid n \in \mathbb{N} \right\} \]

is uniformly equicontinuous.
8. Assume that the improper integral \( \int_0^\infty \frac{f(x)}{x} \, dx \) exists and show that

\[
\int_0^\infty \frac{f(xy)}{x} \, dx = \int_0^\infty \frac{f(x)}{x} \, dx \quad \forall y \in (0, \infty).
\]
9. Let \((f_n)_{n \in \mathbb{N}}\) be a decreasing sequence of real-valued functions on \([a, b]\) which converges uniformly to \(f_\infty \equiv 0\). Show that
\[
\sum_{n=1}^{\infty} (-1)^n f_n
\]
converges uniformly.