Math 205 Winter Term 2006

Final Examination

Print your name: ________________ ________________

Print your ID #: ________________________________

You have 2 hours to solve the problems. Good luck!

1. Let $g \in \mathrm{C}^1\big([c,d],[a,b]\big)$ and $f \in \mathrm{C}\big([a,b]\big)$. Define

$$F(x) := \int_a^{g(x)} f(y)\,dy\,,\ x \in (a,b)\,.$$

Show that F is differentiable and compute its derivative.

2. Let $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $[0,\infty)$ and prove that

$$\sum x_n < \infty \iff \sum 2^k x_{2^k} < \infty\,.$$

3. Let a sequence of functions $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(x) = \cos(x)^n \,, \; x \in [0, \frac{\pi}{2}] \,, \; n \in \mathbb{N} \,.$$

Let $g \in \mathrm{C}\big([0, \frac{\pi}{2}]\big)$ be such that $g(0) = 0$. What is the limit of $(gf_n)_{n\in\mathbb{N}}$? Is the convergence pointwise? Is it uniform? Justify your answer.

4. Show that f defined through

$$f(x) = \log^2(1 + x)$$

is analytic in a neighborhood of the origin. Compute the coefficients of its power series expansion about $x = 0$.

5. Let (M, d) be a metric space. For a subset $A \subset M$ define

$$\bar{A} = A \cup \mathrm{LP}(A)$$

and show that

$$\bar{A} = \bigcap \{B \subset M \mid A \subset B \text{ and } B \text{ is closed}\}\,.$$

6. Let (M, d) be a metric space. For a subset $A \subset M$ define

$$\mathring{A} := \{x \in A \,|\, \exists\, r > 0 \text{ s.t. } \mathbb{B}(x, r) \subset A\}\,.$$

Prove or disprove: $(A \cup B)^\circ = \mathring{A} \cup \mathring{B}$, $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$

7. Prove or disprove:

$$\{\frac{1}{\sqrt{n}}\tanh(nx) : \mathbb{R} \to \mathbb{R} \,|\, n \in \mathbb{N}\}$$

is uniformly equicontinuous.

8. Assume that the improper integral $\int_0^\infty \frac{f(x)}{x}\, dx$ exists and show that

$$\int_0^\infty \frac{f(xy)}{x}\, dx = \int_0^\infty \frac{f(x)}{x}\, dx \;\forall\, y \in (0, \infty)\,.$$

9. Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real-valued functions on $[a, b]$ which converges uniformly to $f_\infty \equiv 0$. Show that

$$\sum_{n=1}^{\infty}(-1)^n f_n$$

converges uniformly.