Final Examination

Print your name: ___________________ ___________________
Print your ID #: ________________________________

You have 2 hours to solve the problems. Good luck!
1. Let $A \in \mathbb{R}^{n \times n}$ and show that the map 
\[ \Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x, y) \mapsto y^T Ax \]
is differentiable and compute its derivative.

2. Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and assume that it is convex, that is, that 
\[ f((1-t)x + ty) \leq (1-t)f(x) + tf(y) \quad \forall x, y \in \mathbb{R}^n \quad \forall t \in [0, 1]. \]
Prove that $D^2 f(x) \geq 0$ (positive definite) for every $x \in \mathbb{R}^n$.

3. Show that the system 
\[
\begin{cases}
  e^{x+y+c} = 1 \\
  \frac{1}{1 + (x-1)^2 + y^2} = d + \frac{1}{2}
\end{cases}
\]
has a unique small solution for every small $c, d \in \mathbb{R}$.

4. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite. Show that 
$M_{n-1} := \{x \in \mathbb{R}^n \mid x^T Ax = 1\}$
is a $(n-1)$-dimensional $C^1$-manifold in $\mathbb{R}^n$.

5. Compute the volume of the set 
$C := \{(x, y, z) \mid 0 \leq x^2 + y^2 \leq z, \quad 0 \leq z \leq 1\}$.

6. Find maxima and minima of the function $f(x, y, z) = 4y - 2z$ on the curve determined by 
$2x - y - z = 2, \quad x^2 + y^2 = 1$.

7. Compute the integral 
\[ \int_0^1 \int_{-3}^3 e^{x^2} \, dy \, dx. \]

8. Find maxima and minima of the function 
$f(x, y) = x^2 e^{-x^2 - y^2}, \quad (x, y) \in \mathbb{R}^2 \setminus (0, 1) = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}.$
Indicate which maxima and minima are strict and which are not.

9. Can the surface parametrized by 
$(s, t) \mapsto (s^3 + t^3, st, s^3 - t^3), \quad \mathbb{R}^2 \to \mathbb{R}^3$
be represented as the graph of a function? If your answer is no, explain why. If it is yes, determine the function.