Assignment 22

1. Let \( f, g \in C^1(\mathbb{R}^2, \mathbb{R}) \) be given. Find necessary conditions for the existence of \( u \in C^2(\mathbb{R}^2, \mathbb{R}) \) such that
\[
\nabla u = (f, g).
\]
Show that those conditions are also sufficient and derive a formula describing \( u \) in terms of \( f \) and \( g \).

2. Consider the system
\[
\begin{align*}
x^2 + y^2 - u^2 - xv &= 0 \\
x y + uv - v^2 &= 0
\end{align*}
\]
and determine the points \((x, y, u, v) \in \mathbb{R}^4\) where the system can be solved for \((u, v)\) for sure.

3. Assume that \( A \in \mathbb{R}^{n \times n} \) and \( \|A\| < 1 \). Show that
\[
\sum_{n=0}^{\infty} A^n
\]
is norm-convergent and that its value is \((\text{id}_{\mathbb{R}^n} - A)^{-1}\). Use this to prove that \( \mathcal{GL}_n(\mathbb{R}) \subset \mathbb{R}^{n \times n} \).

4. Prove that there is a neighborhood \( U \) of \( \text{id}_{\mathbb{R}^n} \) in which the equation
\[
X^2 = M
\]
has a unique solution \( X \) close to the identity for each \( M \in U \). Compute the derivative of
\[
f : U \to \mathbb{R}^{n \times n}, \ M \mapsto X =: \sqrt{M}.
\]

5. Let \( F \in C^2(D, \mathbb{R}^n) \) for some \( D \subset \mathbb{R}^n \). In order to solve \( F(x) = 0 \) one can use Newton iteration
\[
x_{n+1} = x_n - DF(x_n)^{-1}F(x_n),
\]
with some initial guess \( x_0 \in D \). Let \( z \in D \) be a non-critical zero of \( F \) and prove that \((x_n)_{n \in \mathbb{N}}\) converges to \( z \) if the initial guess is chosen close enough to \( z \).

Homework due by Thursday, May 18 2006