Assignment 22

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1. Let $f, g \in \mathrm{C}^1(\mathbb{R}^2, \mathbb{R})$ be given. Find necessary conditions for the existence of $u \in \mathrm{C}^2(\mathbb{R}^2, \mathbb{R})$ such that
$$\nabla u = (f, g)\,.$$
Show that those conditions are also sufficient and derive a formula describing u in terms of f and g.

2. Consider the system
$$\begin{cases} x^2 + y^2 - u^2 - xv = 0 \\ xy + uv - v^2 = 0 \end{cases}$$
and determine the points $(x, y, u, v) \in \mathbb{R}^4$ where the system can be solved for (u, v) for sure.

3. Assume that $A \in \mathbb{R}^{n \times n}$ and $\|A\| < 1$. Show that
$$\sum_{n=0}^{\infty} A^n$$
is norm-convergent and that its value is $(\mathrm{id}_{\mathbb{R}^n} - A)^{-1}$. Use this to prove that $\mathcal{GL}_n(\mathbb{R}) \overset{o}{\subset} \mathbb{R}^{n \times n}$.

4. Prove that there is a neighborhhod U of $\mathrm{id}_{\mathbb{R}^n}$ in which the equation
$$X^2 = M$$
has a unique solution X close to the identity for each $M \in U$. Compute the derivative of
$$f : U \to \mathbb{R}^{n \times n}\,,\ M \mapsto X =: \sqrt{M}\,.$$

5. Let $F \in \mathrm{C}^2(D, \mathbb{R}^n)$ for some $D \overset{o}{\subset} \mathbb{R}^n$. In order to solve $F(x) = 0$ one can use Newton iteration
$$x_{n+1} = x_n - DF(x_n)^{-1} F(x_n)\,,$$
with some initial guess $x_0 \in D$. Let $z \in D$ be a non-critical zero of F and prove that $(x_n)_{n \in \mathbb{N}}$ converges to z if the initial guess is chosen close enough to z.

Homework due by Thursday, May 18 2006