Print your name: ________________ ________________

Print your ID #: ________________________________

You have 50 minutes to solve the problems. Good luck!
1. Let $M$ be a complete metric space and $f : M \to M$ a contraction. Denoting by $x_0$ the fixed point of $f$, prove that

$$d(x, x_0) \leq \frac{1}{1 - r}d(x, f(x))$$

for any $x \in M$ where $r \in (0, 1)$ is the Lipschitz constant of $f$. 
2. Let $f, g \in C^1(\mathbb{R}^n, \mathbb{R})$ be positive functions. Show that $fg \in C^1(\mathbb{R}^n, \mathbb{R})$ and compute $D(fg)$. Show that, if $fg$ attains a minimum at $x$, then $\nabla f(x)$ and $\nabla g(x)$ are linearly dependent.
3. Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$ and assume that

$$x \in L := f^{-1}(5) := \{y \in \mathbb{R}^n \mid f(y) = 5\}.$$ 

If $\gamma \in C^1((0, 1), L)$ is a curve through $x$, show that $\nabla f(x)$ is orthogonal to the curve $\gamma$ at $x$. 
4. Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$ with $D^2 f(x) > 0$ for some $x \in \mathbb{R}^n$. Show that, in a neighborhood of $x$, the graph

$$G_f := \{(x, f(x)) \mid x \in \mathbb{R}^n\}$$

of $f$ lies above its tangent plane at $x$. 
5. Let $f \in C([a, b], \mathbb{R})$ and $g \in C([c, d], \mathbb{R})$. Show that

$$\int_R f(x)g(y) \, d(x, y) = \left[ \int_a^b f(x) \, dx \right] \left[ \int_c^d g(x) \, dx \right]$$

for $R := [a, b] \times [c, d]$. 