Midterm Examination - Solutions

1. Let M be a complete metric space and $f : M \to M$ a contraction. Denoting by x_0 the fixed point of f, prove that

$$d(x, x_0) \leq \frac{1}{1-r} d(x, f(x))$$

for any $x \in M$ where $r \in (0,1)$ is the Lipschitz constant of f.
Solution: Since $f(x_0) = x_0$ we see that

$$d(f(x), x_0) = d(f(x), f(x_0)) \leq r\, d(x, x_0)$$

and, therefore, by the triangular inequality

$$d(x, x_0) \leq d(x, f(x)) + d(f(x), x_0) \leq d(f(x), x) + r d(x, x_0)$$

which readily implies the claim.

2. Let $f, g \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R})$ be positive functions. Show that $fg \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R})$ and compute $D(fg)$. Show that, if fg attains a minimum at x, then $\nabla f(x)$ and $\nabla g(x)$ are linearly dependent.
Solution: By assumption we have that both f and g possess continuous partial derivatives in all direction. Then

$$\partial_j (fg)(x) = f(x)\partial_j g(x) + g(x)\partial_j f(x)\,,\; j = 1, \dots, n$$

and thus all partial derivatives of fg exist and are continuous. We conclude that fg is differentiable and, by the above formula for its partial derivatives, we infer that

$$\nabla(fg)(x) = f(x)\nabla g(x) + g(x)\nabla f(x)\,.$$

At a point of minimum we would have that

$$0 = \nabla(fg)(x) = f(x)\nabla g(x) + g(x)\nabla f(x)$$

which would make the gradients linearly dependent since f and g never vanish.

3. Let $f \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R})$ and assume that

$$x \in L := f^{-1}(5) := \{y \in \mathbb{R}^n \,|\, f(y) = 5\}\,.$$

If $\gamma \in \mathrm{C}^1\big((0,1), L\big)$ is a curve through x, show that $\nabla f(x)$ is orthogonal to the curve γ at x.
Solution: Let γ be a curve through x with the above properties and say that $\gamma(0.5) = x$. Then

$$f\big(\gamma(t)\big) = 5 \,\forall\, t \in (0,1)$$

and therefore, by taking one derivative,

$$\nabla f\big(\gamma(t)\big) \cdot \dot\gamma(t) = 0 \,\forall\, t \in (0,1)\,.$$

In particular we have that

$$\nabla f(x) \cdot \dot\gamma(0.5) = 0$$

which gives the desired result since $\dot\gamma(0.5)$ is clearly tangent to the curve at x.

4. Let $f \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R})$ with $D^2 f(x) > 0$ for some $x \in \mathbb{R}^n$. Show that, in a neighborhood of x, the graph

$$G_f := \{(x, f(x)) \,|\, x \in \mathbb{R}^n\}$$

of f lies above its tangent plane at x.
Solution: Since $D^2 f(x)$ is positive definite and $D^2 f$ is continuous, we can find $\delta > 0$ such that $D^2 f(y)$ is still positive if $|y - x|_2 \le \delta$. Then, by Taylor expansion with remainder, we obtain that

$$f(y) = \underbrace{f(x) + Df(x)(y-x)}_{\text{equation for tangent plane}} + \frac{1}{2}(y-x)^T D^2 f(z)(y-x)$$

for any $y \in \mathbb{B}(x, \delta)$ and some z on the segment between x and y. Thus the claim follows since

$$\frac{1}{2}(y-x)^T D^2 f(z)(y-x) > 0$$

regardless of $y, z \in \mathbb{B}(x, \delta)$.
Alternatively, consider the function

$$g : \mathbb{R}^n \to \mathbb{R}\,,\ y \mapsto f(y) - f(x) - Df(x)(y-x)\,.$$

If this map is positive in a neighborhood of x, the claim follows. But g has a minimum at x where $g(x) = 0$ since

$$Dg(x) = 0 \text{ and } D^2 f(x) > 0 \text{ by assumption.}$$

5. Let $f \in \mathrm{C}([a,b],\mathbb{R})$ and $g \in \mathrm{C}([c,d],\mathbb{R})$. Show that

$$\int_R f(x)g(y)\,d(x,y) = [\int_a^b f(x)\,dx][\int_c^d g(x)\,dx]$$

for $R := [a,b] \times [c,d]$.
Solution: Since f and g are integrable, we can find, for given $\varepsilon > 0$, partitions $P_x \in \mathcal{P}(a,b)$ and $P_y \in \mathcal{P}(c,d)$ such that

$$|S_f(P_x) - I_f| \le \frac{\varepsilon}{I_f + I_g + 1}\,,$$
$$|S_g(P_y) - I_g| \le \frac{\varepsilon}{I_f + I_g + 1}\,,$$

where $I_f = \int_a^b f(x)\,dx$ and $I_g = \int_c^d g(x)\,dx$. Then

$$\Big|\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} f(\xi_i)g(\eta_j)(\xi_i - \xi_{i-1})(\eta_j - \eta_{j-1}) - I_f I_g\Big|$$
$$\le |\sum_{i=1}^{N_x} f(\xi_i)(\xi_i - \xi_{i-1})||S_g(P_y)| + |I_f||\sum_{j=1}^{N_y} g(\eta_j)(\eta_j - \eta_{j-1})| \le \varepsilon\,.$$

Another more direct proof is based on

$$\int_R F(x,y)\,d(x,y) = \int_a^b \int_c^d F(x,y)\,dy\,dx\,,$$

which we proved in class, with $F(x,y) = f(x)g(y)$.