1. Assume that $f \in C([0,1])$ and that 

$$F \in C([0,1] \times [0,1] \times \mathbb{R}), \quad \partial_u F \in C([0,1] \times [0,1] \times \mathbb{R})$$

and consider the integral equation

$$u(x) = \int_0^1 F(x, y, u(y)) \, dy + f(x), \quad 0 \leq x \leq 1.$$ 

Show that, if $\|\partial_u F\|_{\infty} < 1$, the integral equation has a unique solution $u \in C([0,1])$.

2. Let $f \in C(B(0,1), \mathbb{R}^n)$ be such that 

$$|f(x)| \leq 1$$

for $|x| = 1$.

Show that $f$ has a fixed point in $B(0,1)$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. Prove that the equation

$$
\begin{cases}
-\Delta u = e^{-u} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

possesses a solution.

4. (Kolmogoroff) Show that a subset $K \subset L_p(\mathbb{R}^n)$ ($1 \leq p < \infty$) is compact iff

(i) $K$ is closed and bounded.

(ii) $\int_{|x| \geq N} |f(x)|^p \, dx \to 0$, $N \to \infty$, uniformly in $f \in K$.

(iii) $\int_{\mathbb{R}^n} |f(x + h) - f(x)|^p \, dx \to 0$, $|h| \to 0$, uniformly in $f \in K$.

[Hint: Use the density of test functions in $L_p(\mathbb{R}^n)$, the strong continuity of the translation semigroup on $L_p(\mathbb{R}^n)$ and Arzela-Ascoli.]

5. Let $1 \leq p < \infty$ and prove that

$$\left( \int_{\mathbb{R}^n} |u(x + h) - u(x)|^p \, dx \right)^{1/p} \leq |h| \|[u]_{1,p}$$

for $u \in W_{p}^{1}(\mathbb{R}^n)$. Use this estimate and Kolmogoroff’s characterization of compactness to show that $W_{p}^{1}(\Omega) \hookrightarrow L_p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open and bounded.

Homework due by Friday, April 9 2010.