1. Let \( A \) be the general elliptic second order differential operator in divergence form on a bounded domain \( \Omega \) with smooth boundary, that is,
\[
A u = -\nabla \cdot (A \nabla u) + (b | \nabla u) + cu
\]
where the coefficients satisfy the assumptions of Chapter 8 of class. Let \( \Phi \in \text{Diff}^2(\Omega, \tilde{\Omega}) \), that is, \( \Phi \) is invertible and \( \Phi \in C^2(\Omega, \tilde{\Omega}) \), \( \Psi := \Phi^{-1} \in C^2(\tilde{\Omega}, \Omega) \).

Letting \( y := \Phi(x) \) and define \( \tilde{u}(y) := u(\Psi(y)) \), compute the operator \( \tilde{A} \) in the new variables, that is the operator satisfying
\[
\tilde{A} \tilde{u} = \tilde{A} \tilde{u}.
\]

2. Prove that the Neumann problem
\[
\begin{align*}
-\Delta u &= f \text{ in } \Omega, \\
\partial_\nu u &= 0 \text{ on } \partial \Omega
\end{align*}
\]
on a bounded domain with smooth boundary has a solution if and only if \( \int_{\Omega} f \, dx = 0 \).

3. Let \( \mathbb{H}^n \) be the upper half-space. Given \( m \in \mathbb{N} \) construct an extension operator \( \text{ext} : C^m(\mathbb{H}^n) \to C^m(\mathbb{R}^n) \) such that
\[
(\text{ext} u)|_{\mathbb{H}^n} = u, \quad u \in C^m(\mathbb{H}^n)
\]

4. Let \( f : \Omega \to \mathbb{R} \) be measurable. Define
\[
\mu_f(t) := \left| \{ x \in \Omega : |f(x)| > t \} \right|
\]
Let \( p > 0 \) and assume \( f \in L_p(\Omega) \). Prove that
\[
\mu_f(t) \leq t^{-p} \|f\|_p^p
\]
and that
\[
\|f\|_p^p = p \int_0^\infty t^{p-1} \mu_f(t) \, dt.
\]

5. Let \( 1 \leq q < r < \infty \) and \( T : L_q(\Omega) \cap L_r(\Omega) \to L_q(\Omega) \cap L_r(\Omega) \) be a linear operator such that
\[
\mu_{Tf}(t) \leq (T_1 \|f\|_q/t)^q \text{ and } \mu_{Tf}(t) \leq (T_2 \|f\|_r/t)^r
\]
for some constants \( T_1 \) and \( T_2 \). Then \( T \) can be extended to an operator \( T \in \mathcal{L}(L_p(\Omega)) \) for any \( p \in (q, r) \) and
\[
\|Tf\|_p \leq c T_1^\alpha T_2^{1-\alpha} \|f\|_p, \quad f \in L_q(\Omega) \cap L_r(\Omega)
\]
where \( 1/p = (1 - \alpha)/r + \alpha/q \).
[Hint: For \( s > 0 \) use \( f = f\chi_{|f|>s} + f\chi_{|f|\leq s} = f_1 + f_2 \) to prove that
\[
\mu_{Tf}(t) \leq \mu_{Tf_1}(t/2) + \mu_{Tf_2}(t/2)
\]
and then use the previous problem and Fubini’s theorem.]

Homework due by Friday, January 15 2010