1. Let $H(t, x)$ be the heat kernel (the fundamental solution for the heat equation introduced in class) and show that $T$ defined through

$$
\begin{cases}
(T(t)u)(x) := \int_{\mathbb{R}^n} H(t, x-y)u(y) \, dy, \ x \in \mathbb{R}^n \\
T(0)u := u
\end{cases}
$$

is a $C_0$-semigroup of contractions on $L_2(\mathbb{R}^n)$ but NOT on $L_\infty(\mathbb{R}^n)$.

A $C_0$-semigroup $T$ on a Banach space $E$ is called analytic if it allows for an analytic strongly continuous extension to a sector $\Sigma_\delta = \{ \arg(z) < \delta \}$ of the complex plane for some $\delta \in (0, \pi/2]$, that is, if

(i) $T(0) = \text{id}_E$, $T(z_1 + z_2) = T(z_1)T(z_2)$, $z_1, z_2 \in \Sigma_\delta$.

(ii) $T : \Sigma_\delta \to \mathcal{L}(E)$ is analytic.

(iii) $\lim_{\Sigma_\delta \ni z \to 0} T(z)x = x$ for all $x \in E$.

It can be shown that the above conditions are equivalent to

(i) $T(t)E \subset \text{dom}(A)$, $t > 0$.

(ii) $\|tAT(t)\|_{\mathcal{L}(E)} \leq c < \infty$, $t > 0$.

where $-A : \text{dom}(A) \subset E \to E$ is the generator of $T$. Show that the $C_0$-semigroup of problem 1 is analytic.

2. Let $-A : \text{dom}(A) \subset E \to E$ be the generator of an analytic $C_0$-semigroup $T$ on $E$. Let $f \in C^\rho([0, T], E)$ for some $\rho \in (0, 1)$ and show that the mild solution $u : [0, T] \to E$ of

$$
\dot{u} + Au = f(t), \ u(0) = x \in E,
$$

given by

$$
u(t) = T(t)x + \int_0^t T(t-\tau)f(\tau) \, d\tau, \ t \in [0, T]\n$$

is actually differentiable for $t > 0$.

3. Let $A : \text{dom}(A) \subset E \to E$ be defined through

$$
E = L_2(0, 1), \quad \text{dom}(A) = \{ u \in H^2(0, 1) \mid u(0) = u(1) = 0 \}, \quad Au = -\partial_{xx} u, \ u \in \text{dom}(A),
$$

and show that $-A$ generates an analytic $C_0$-semigroup on $E$. 
4. Define $T(t) \in \mathcal{L}(L_2(\mathbb{R}^n))$ through
\[
\begin{cases}
(1 - \Delta)^{-t} = \mathcal{F}^{-1}(1 + |\xi|^2)^{-t}\mathcal{F}, & t > 0 \\
\text{id}_{L_2(\mathbb{R}^n)}, & t = 0.
\end{cases}
\]
Show that $T$ is a $C_0$-semigroup on $L_2(\mathbb{R}^n)$. What is its generator?

5. For a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, for $b \in L_\infty(\Omega)$ and $c \in L_\infty(\Omega)$ let $A$ be the operator induced by the Dirichlet form
\[
a(u,v) = \int_\Omega \left( (\nabla u | \nabla v) + (b | \nabla u)v + cuv \right) dx, \quad u, v \in H^1(\Omega)
\]
on $H^{-1}(\Omega)$. Show that it generates a $C_0$-semigroup on $H^{-1}(\Omega)$.

The Homework is due Monday, January 25 2010