

Assignment 1

1. Show that $v.p.\frac{1}{x}$ is a well-defined distribution. Recall that

$$\langle v.p.\frac{1}{x}, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} dx$$

for any test function $\varphi \in \mathcal{D}(\mathbb{R})$. Then prove that

$$u_\varepsilon^\pm \rightarrow \mp i\pi\delta + v.p.\frac{1}{x} \text{ as } \varepsilon \downarrow 0$$

for $u_\varepsilon^\pm(x) := \frac{1}{x \pm i\varepsilon}$, $x \in \mathbb{R}$, in the sense of distributions.

2. Compute f' and f'' for $f(x) = \log|x|$, $x \in \mathbb{R}$ and $f(x) = |x|$, $x \in \mathbb{R}$, respectively, in the sense of distributions.

3. Let $\rho : \mathbb{R}^n \mapsto \mathbb{R}$ be an integrable function with

$$\text{supp}(\rho) \subset \mathbb{B}(0, 1) \text{ and } \int_{\mathbb{R}^n} \rho(x) dx = 1.$$

Show that $\rho_\varepsilon \rightarrow \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$ for

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n.$$

4. Consider the function u defined as

$$u(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, (x, y) \in \mathbb{R} \times (0, \infty)$$

Then $u \in C^\infty(\mathbb{R} \times (0, \infty))$. Compute $\lim_{y \rightarrow 0} u(\cdot, y)$.

5. Given a continuous function $f \in C(\Omega)$ with $\text{supp}(f) \subset\subset \Omega$, show that a sequence $(\varphi_j)_{j \in \mathbb{N}}$ of test functions in $\mathcal{D}(\Omega)$ can be found such that

$$\|\varphi_j - f\|_\infty := \sup_{x \in \Omega} |\varphi_j(x) - f(x)| \rightarrow 0 \text{ as } j \rightarrow \infty.$$