1. Let $0 \neq \alpha \in \mathbb{R}$ and consider the initial value problem for Euler’s equation

$$\sum_{k=1}^{n} x_k \partial_{x_k} u = \alpha u, \ u(x_1, \ldots, x_{n-1}, 1) = g(x_1, \ldots, x_{n-1}).$$

Show that its solution satisfies the functional equation

$$u(\lambda x) = \lambda \alpha u(x), \ x \neq 0, \ \lambda > 0.$$  

What is the behavior of the solution at $x = 0$?

2. Consider the quasilinear initial value problem

$$\begin{cases} u_t + uu_x = 0, & (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, x) = g(x), & x \in \mathbb{R} \end{cases}$$

Solve the equation and analyze the possible onset of singularities. What conditions on $g$ would prevent the solution from developing singularities?

3. Let $u \in C^1(\mathbb{B}(0, 1))$ be a solution of

$$a(x, y) u_x + b(x, y) u_y = -u$$

and assume that $a(x, y)x + b(x, y) y > 0$ for $(x, y) \in \mathbb{S}^1$. Show that $u \equiv 0$ then.

4. Consider the equation

$$\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0.$$  

Any function $u$ with

$$\int_{\mathbb{R}^2} \left[ R(u) \phi_y + S(u) \phi_x \right] d(x, y) = 0, \ \phi \in C^\infty_0(\mathbb{R}^2)$$

is called weak solution. Assume that $u$ is continuously differentiable away from some curve parametrized by $(s(y), y), y \in \mathbb{R}$, across which it has a jump discontinuity. Conclude that

$$s'(y) = \frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)}$$

where $u^\pm$ indicate the one sided limits of $u$ approaching the curve.
5. Consider the eikonal equation

\[ c^2(u_x^2 + u_y^2) = 1 \]

Let \( \gamma_t \) be the level line \( [u(x, y) = t] \) of a solution \( u \). Show that a point \( (x, y) \) moves in a direction perpendicular to \( \gamma_t \) at constant speed \( c \) (along a characteristic).