Assignment 13

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1. Find the unique weak solutions of
$$\begin{cases} u_t + \frac{1}{2}|\nabla u|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty)\,, \\ u(0,\cdot) = \pm|\cdot| & \text{on } \mathbb{R}^n\,. \end{cases}$$

2. Consider the following *porous medium equation*:
$$u_t - \triangle(u^\gamma) = 0 \text{ in } \mathbb{R}^n \times (0,\infty)\,.$$
(i) Show that there exists a solution of the form $u(t,x) = v(t)w(x)$.
(ii) Find a solution of the form $u(t,x) = \frac{1}{t^\alpha} v\big(\frac{x}{t^\beta}\big)$.
What are the main features of these solutions?

3. Find a transformation $w = \Phi(u)$ which reduces the nonlinear equation
$$\begin{cases} u_t - a\triangle u + b|\nabla u|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty)\,, \\ u = g & \text{in } \mathbb{R}^n \times \{0\} \end{cases}$$
to a linear one and use the latter to find a solution of the former.

4. Use the solution to the previous problem to find a solution for the *viscous Burger's equation*
$$\begin{cases} u_t - a u_{xx} + u\, u_x = 0 & \text{in } \mathbb{R} \times (0,\infty)\,, \\ u = g & \text{in } \mathbb{R} \times \{0\}\,. \end{cases}$$
[Hint: Derive an equation for $w(t,x) = \int_{-\infty}^x u(t,y)\,dy$.]

5. Consider *Euler's equation*
$$\begin{cases} u_t - u \cdot Du = -\nabla p & \text{in } \mathbb{R}^3 \times (0,\infty)\,, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty)\,, \\ u = g & \text{in } \mathbb{R}^3 \times \{0\}\,. \end{cases}$$
Assume that there exists v such that $u = \nabla v$ and derive a simpler equation for v. Once v is obtained, how can p be derived?

Homework due Friday, April 12 2013