

## Assignment 15

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Let  $\Omega \subset \mathbb{R}^n$  be open. A map  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is called *Carathéodory function* whenever

- (i)  $f(\cdot, s) : \Omega \rightarrow \mathbb{R}$  is measurable for every  $s \in \mathbb{R}$ .
- (ii)  $f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is continuous for almost every  $x \in \Omega$ .

1. Let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function and  $p, q \geq 1$ . Assume that

$$|f(x, s)| \leq c|s|^{p/q} + g(x)$$

for some  $g \in L_q(\Omega)$ . Prove that the Nemytzki operator (substitution operator)  $N_f : L_p(\Omega) \rightarrow L_q(\Omega)$  defined through

$$(N_f u)(x) := f(x, u(x)), \quad x \in \Omega$$

is well-defined, continuous and maps bounded sets onto bounded sets.

2. Let  $\Omega \subset \mathbb{R}^n$  be open and bounded. Assume that the Carathéodory function  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$\underline{f} \leq \frac{f(x, u) - f(x, v)}{u - v} \leq \bar{f} \text{ and that } f(\cdot, 0) \in L_2(\Omega)$$

with  $\sigma(-\Delta_D) \cap [\underline{f}, \bar{f}] = \emptyset$ . Show that

$$\begin{cases} \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

possesses a unique weak solution  $u \in \mathring{H}^1(\Omega)$ .

3. Let  $H$  be a Hilbert space. Prove that

$$x_n \rightarrow x, y_n \rightarrow y (n \rightarrow \infty) \Rightarrow (x_n | y_n) \rightarrow (x | y) (n \rightarrow \infty).$$

4. Let  $E$  be a normed vector space and  $A \in \mathcal{L}(E)$  a compact operator. Show that

$$\dim(\ker[(\lambda - A)^n]) < \infty \text{ for any } \lambda \neq 0 \text{ and } n = 1, 2, \dots$$

[Use the fact that  $\overline{\mathbb{B}}_E(0, 1)$  is compact iff  $\dim(E) < \infty$  and that eigenvectors to different eigenvalues are linearly independent.]

5. Let  $K \in \mathcal{L}(H)$  be a compact operator. Show that it maps weakly convergent sequences to convergent sequences.