

Assignment 5

1. Compute a fundamental solution G_2 for the wave operator $\partial_t^2 - \Delta$ on $\mathbb{R} \times \mathbb{R}^2$. [Hint: Let G_3 be the fundamental solution for the wave operator on $\mathbb{R} \times \mathbb{R}^3$ introduced in class and show that

$$\langle G_3(t, x, x_3), \varphi(t, x) \mathbf{1}(x_3) \rangle$$

can be made sense of for $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$. Define G_2 through

$$\langle G_2, \varphi \rangle = \langle G_3, \varphi \mathbf{1}(x_3) \rangle = \frac{1}{4\pi} \int_0^\infty \frac{1}{t} \int_{\mathbb{S}_t^2} \varphi(t, x) d\sigma_{\mathbb{S}_t^2}(x) dt$$

and use the fact that φ is independent of x_3 to simplify the integral.]

2. A fundamental solution $G(\cdot, y)$ for the Laplacian $-\Delta$ on a domain $\Omega \subset \mathbb{R}^n$ with pole at $x = y$ and satisfying $G(\cdot, y)|_{\partial\Omega} = 0$ is called *Green's function* for the Dirichlet problem in Ω (considered as a function of $(x, y) \in \Omega \times \Omega$). Compute a Green's function for the Dirichlet problem in the half-space $\mathbb{H}^n = \mathbb{R}^{n-1} \times (0, \infty)$, $n \geq 2$.
3. Let G be the fundamental solution for the heat operator $\partial_t - \Delta$ on $\mathbb{R} \times \mathbb{R}^n$ introduced in class. Show that

$$G(t, \cdot) \rightarrow \delta, \quad t \rightarrow 0+, \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and find a representation for the solution of the Cauchy problem

$$\partial_t u - \Delta u = f \in L_{1,loc}([0, \infty), \mathcal{S}'), \quad u(0, \cdot) = u_0 \in \mathcal{S}'$$

on $(0, \infty) \times \mathbb{R}^n$ in terms of G .

4. Compute the general solution to the following initial value problem for the wave equation

$$\partial_t^2 u - \partial_x^2 u = f(t, x), \quad u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1,$$

in $(0, \infty) \times \mathbb{R}$ in terms of a fundamental solution for the wave operator. What is the regularity of the solution if

$$(f, u_0, u_1) \in C([0, \infty), \mathcal{S}) \times \mathcal{S} \times \mathcal{S}?$$

5. Let $u : \Omega \rightarrow \mathbb{R}$ be a harmonic function (i.e. $\Delta u = 0$) on the domain Ω . Prove *Gauss's law of the arithmetic mean*

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{|x-y|=r} u(y) d\sigma_r(y)$$

valid for all x and r such that $\mathbb{B}(x, r) \subset \Omega$.

[Hint: Use Green's formula with u and a Green's function for the Dirichlet problem on $\mathbb{B}(x, r)$.]

Homework due by Friday, November 16 2012