1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and prove Poincaré's Inequality

$$\|u\|_{L^p(\Omega)} \leq c \|\nabla u\|_{L^p(\Omega)}, \; u \in W^1_p(\Omega).$$

What does the constant $c$ depend on? Is the boundedness assumption really necessary?

2. Let $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty)$ and define

$$W^m_p(\Omega) = \left\{ u \in D'(\Omega) \mid \partial^\alpha u \in L^p(\Omega), \; |\alpha| \leq m \right\}.$$

Show that $W^m_p(\Omega)$ is a Banach space if endowed with the norm $\|\cdot\|_{m,p}$ defined by

$$\|u\|_{m,p} = \left( \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, \; u \in W^m_p(\Omega).$$

Prove that $W^1_p(0,1) \hookrightarrow BUC^{1-1/p}([0,1])$.

[Hint: Use the fact that $C^1([0,1])$ is dense in $W^1_p(0,1)$]

3. Prove that $S(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$.

4. Let $\Omega = B(0,1/2)$ and the function $u$ be defined through

$$u(x,y) = \log \left( \log \left( \frac{2}{\sqrt{x^2 + y^2}} \right) \right), \; (x,y) \in \Omega.$$

Then $u$ is obviously not continuous in $(x,y) = (0,0)$. Prove that, however, $u \in H^1(\Omega)$. Let now

$$u(x,y) = xy \left[ \log \left| \log |(x,y)| \right| - \log \log 2 \right], \; (x,y) \in \Omega.$$

Show that

$$u \in C^1(\bar{\Omega})$$

and

$$\partial^2_j u \in C(\bar{\Omega}), \; j = 1,2$$

is a solution of the Dirichlet problem in the ball with continuous datum but $u \not\in C^2(\bar{\Omega})$.

5. Let $E$ be a Banach space and $A : \text{dom}(A) \subset E \rightarrow E$ a linear, possibly unbounded, operator on $E$. $A$ is said to be invertible if there exists a bounded operator $B \in \mathcal{L}(E)$ such that

$$AB = \text{id}_E \; \text{and} \; BA = \text{id}_{\text{dom}(A)}.$$

Such an operator $A$ can fail to be invertible either because it has non trivial kernel (not injective)

$$\ker(A) \neq \{0\}.$$
or because it is not surjective
\[ R(A) \neq E \]
but, also, because its “inverse” is unbounded.

Let
\[
E = l_2(\mathbb{N}) := \{(x_j)_{j \in \mathbb{N}} \mid x_j \in \mathbb{R} \ \forall \ j \in \mathbb{N} \text{ and } \sum_{j=1}^{\infty} x_j^2 < \infty \}
\]
with the norm naturally induced by the scalar product
\[
(x|y) = \sum_{j=1}^{\infty} x_j y_j, \ x, y \in l_2(\mathbb{N}).
\]

For each one of the ways described find an operator \(A\) on \(l_2(\mathbb{N})\) which fails to be invertible in that and no other way. In general the set
\[
\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda - A \text{ is not invertible} \} \subset \mathbb{C}
\]
is called spectrum of \(A\). Show that it is a closed set.