Assignment 6

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and prove *Poincaré's Inequality*
$$\|u\|_{\mathrm{L}_p(\Omega)} \leq c\|\nabla u\|_{\mathrm{L}_p(\Omega)}\,,\ u \in \mathring{\mathrm{W}}^1_p(\Omega)\,.$$
What does the constant c depend on? Is the boundedness assumption really necessary?

2. Let $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty)$ and define
$$\mathrm{W}^m_p(\Omega) = \big\{u \in \mathcal{D}'(\Omega)\,\big|\,\partial^\alpha u \in \mathrm{L}_p(\Omega)\,,\ |\alpha| \leq m\big\}\,.$$
Show that $\mathrm{W}^m_p(\Omega)$ is a Banach space if endowed with the norm $\|\cdot\|_{m,p}$ defined by
$$\|u\|_{m,p} = \Big(\sum_{|\alpha|\leq m} \|\partial^\alpha u\|^p_{\mathrm{L}_p(\Omega)}\Big)^{1/p}\,,\ u \in \mathrm{W}^m_p(\Omega)\,.$$
Prove that $\mathrm{W}^1_p(0,1) \hookrightarrow \mathrm{BUC}^{1-1/p}([0,1])$.
[Hint: Use the fact that $\mathrm{C}^1([0,1])$ is dense in $\mathrm{W}^1_p(0,1)$]

3. Prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathrm{H}^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$.

4. Let $\Omega = \mathbb{B}(0, 1/2)$ and the function u be defined through
$$u(x,y) = \log\big(\log(\frac{2}{\sqrt{x^2+y^2}})\big)\,,\ (x,y) \in \Omega\,.$$
Then u is obviously not continuous in $(x, y) = (0, 0)$. Prove that, however, $u \in H^1(\Omega)$. Let now
$$u(x,y) = xy\,\big[\log\big|\log|(x,y)|\big| - \log\log 2\big]\,,\ (x,y) \in \Omega.$$
Show that
$$u \in \mathrm{C}^1(\bar{\Omega}) \text{ and } \partial^2_j u \in \mathrm{C}(\bar{\Omega})\,,\ j = 1, 2$$
is a solution of the Dirichlet problem in the ball with continuous datum but $u \notin \mathrm{C}^2(\bar{\Omega})$.

5. Let E be a Banach space and $A : \mathrm{dom}(A) \subset E \longrightarrow E$ a linear, possibly unbounded, operator on E. A is said to be invertible if there exists a bounded operator $B \in \mathcal{L}(E)$ such that
$$AB = \mathrm{id}_E \ \text{ and } BA = \mathrm{id}_{\mathrm{dom}(A)}\,.$$
Such an operator A can fail to be invertible either because it has non trivial kernel (not injective)
$$\ker(A) \neq \{0\}$$

or because it is not surjective

$$\overline{R(A)} \neq E$$

but, also, because its "inverse" is unbounded.
Let

$$E = l_2(\mathbb{N}) := \Big\{ (x_j)_{j\in\mathbb{N}} \,|\, x_j \in \mathbb{R}\, \forall j \in \mathbb{N} \text{ and } \sum_{j=1}^{\infty} x_j^2 < \infty \Big\}$$

with the norm naturally induced by the scalar product

$$(x|y) = \sum_{j=1}^{\infty} x_j\, y_j \,,\ x, y \in l_2(\mathbb{N})\,.$$

For each one of the ways described find an operator A on $l_2(\mathbb{N})$ which fails to be invertible in that and no other way. In general the set

$$\sigma(A) = \{\lambda \in \mathbb{C} \,|\, \lambda - A \text{ is not invertible}\} \subset \mathbb{C}$$

is called spectrum of A. Show that it is a closed set.

Homework due by Wednesday, November 28 2012