Assignment 1

1. Show that $v.p.\frac{1}{x}$ is a well-defined distribution. Recall that

$$\langle v.p.\frac{1}{x}, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| \ge \varepsilon} \frac{\varphi(x)}{x} dx$$

for any test function $\varphi \in \mathcal{D}(\mathbb{R})$. Then prove that

$$u_{\varepsilon}^{\pm} \to \mp i\pi\delta + v.p.\frac{1}{x}$$
 as $\varepsilon \downarrow 0$

for $u_{\varepsilon}^{\pm}(x):=\frac{1}{x\pm i\varepsilon}\,,\;x\in\mathbb{R}\,,$ in the sense of distributions.

- 2. Compute f' and f'' for $f(x) = \log |x|$, $x \in \mathbb{R}$ and f(x) = |x|, $x \in \mathbb{R}$, respectively, in the sense of distributions.
- 3. Let $\rho: \mathbb{R}^n \to \mathbb{R}$ be an integrable function with

$$\operatorname{supp}(\rho) \subset \mathbb{B}(0,1) \text{ and } \int_{\mathbb{R}^n} \rho(x) dx = 1.$$

Show that $\rho_{\varepsilon} \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$ for

$$\rho_{\varepsilon}(x) := \frac{1}{\varepsilon^n} \rho(\frac{x}{\varepsilon}), \ x \in \mathbb{R}^n.$$

4. Consider the function u defined as

$$u(x,y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, \ (x,y) \in \mathbb{R} \times (0,\infty)$$

Then $u \in C^{\infty}(\mathbb{R} \times (0, \infty))$. Compute $\lim_{y \to 0} u(\cdot, y)$.

5. Given a continuous function $f \in C(\Omega)$ with $\operatorname{supp}(f) \subset\subset \Omega$, show that a sequence $(\varphi_j)_{j\in\mathbb{N}}$ of test functions in $\mathcal{D}(\Omega)$ can be found such that

$$\|\varphi_j - f\|_{\infty} := \sup_{x \in \Omega} |\varphi_j(x) - f(x)| \longrightarrow 0 \text{ as } j \to \infty.$$