Let $\Omega \subset \mathbb{R}^n$ be open. A map $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory function whenever

(i) $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}$.
(ii) $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \Omega$.

1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $p, q \geq 1$. Assume that
   \[ |f(x, s)| \leq c|s|^{p/q} + g(x) \]
   for some $g \in L_q(\Omega)$. Prove that the Nemytski operator (substitution operator) $N_f : L_p(\Omega) \to L_q(\Omega)$ defined through
   \[ (N_fu)(x) := f(x, u(x)), x \in \Omega \]
   is well-defined, continuous and maps bounded sets onto bounded sets.

2. Let $H$ be a Hilbert space. Prove that
   \[ x_n \to x, y_n \rightharpoonup y (n \to \infty) \Rightarrow (x_n|y_n) \to (x|y) (n \to \infty). \]

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. Prove that the equation
   \[
   \begin{cases}
   -\Delta u = e^{-u} & \text{in } \Omega, \\
   u = 0 & \text{on } \partial\Omega,
   \end{cases}
   \]
   possesses a solution.

4. (Kolmogoroff) Show that a subset $K \subset L_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ is compact iff
   (i) $K$ is closed and bounded.
   (ii) $\int_{|x| \geq N} |f(x)|^p \, dx \to 0, N \to \infty$, uniformly in $f \in K$.
   (iii) $\int_{\mathbb{R}^n} |f(x + h) - f(x)|^p \, dx \to 0, |h| \to 0$, uniformly in $f \in K$.
   [Hint: Use the density of test functions in $L_p(\mathbb{R}^n)$, the strong continuity of the translation semigroup on $L_p(\mathbb{R}^n)$ and Arzéla-Ascoli.]

5. Let $1 \leq p < \infty$ and prove that
   \[ \left( \int_{\mathbb{R}^n} |u(x + h) - u(x)|^p \, dx \right)^{1/p} \leq |h| \|u\|_{1,p} \]
   for $u \in W^1_p(\mathbb{R}^n)$. Use this estimate and Kolmogoroff’s characterization of compactness to show that $W^1_p(\Omega) \hookrightarrow L_p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open and bounded.
Homework due by Wednesday, February 18 2015.