1. Compute a fundamental solution $G_2$ for the wave operator $\partial^2_t - \triangle$ on $\mathbb{R} \times \mathbb{R}^2$. [Hint: Let $G_3$ be the fundamental solution for the wave operator on $\mathbb{R} \times \mathbb{R}^3$ introduced in class and show that

$$\langle G_3(t, x, x_3), \varphi(t, x) \mathbf{1}(x_3) \rangle$$

can be made sense of for $\varphi \in S(\mathbb{R} \times \mathbb{R}^2)$. Define $G_2$ through

$$\langle G_2, \varphi \rangle = \langle G_3, \varphi \mathbf{1}(x_3) \rangle = \frac{1}{4\pi} \int_0^\infty \frac{1}{t} \int S^2_t \varphi(t, x) d\sigma_{S^2_t} \, dt$$

and use the fact that $\varphi$ is independent of $x_3$ to simplify the integral.]

2. A fundamental solution $G(\cdot, y)$ for the Laplacian $-\triangle$ on a domain $\Omega \subset \mathbb{R}^n$ with pole at $x = y$ and satisfying $G(\cdot, y)|_{\partial\Omega} = 0$ is called Green’s function for the Dirichlet problem in $\Omega$ (considered as a function of $(x, y) \in \Omega \times \Omega$). Compute a Green’s function for the Dirichlet problem in the half-space $\mathbb{H}^n = \mathbb{R}^{n-1} \times (0, \infty)$, $n \geq 2$.

3. Let $G$ be the fundamental solution for the heat operator $\partial_t - \triangle$ on $\mathbb{R} \times \mathbb{R}^n$ introduced in class. Show that

$$G(t, \cdot) \to \delta, \ t \to 0+, \ \text{in } S'(\mathbb{R}^n)$$

and find a representation for the solution of the Cauchy problem

$$\partial_t u - \triangle u = f \in L_{1,loc}([0, \infty), S'), \ u(0, \cdot) = u_0 \in S'$$

on $(0, \infty) \times \mathbb{R}^n$ in terms of $G$.

4. Compute the general solution to the following initial value problem for the wave equation

$$\partial^2_t u - \partial^2_x u = f(t, x), \ u(0, \cdot) = u_0, \ \partial_t u(0, \cdot) = u_1,$$

in $(0, \infty) \times \mathbb{R}$ in terms of a fundamental solution for the wave operator. What is the regularity of the solution if $(f, u_0, u_1) \in C([0, \infty), S) \times S \times S$?

5. Let $u : \Omega \to \mathbb{R}$ be a harmonic function (i.e. $\Delta u = 0$) on the domain $\Omega$. Prove Gauss’s law of the arithmetic mean

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{|x - y| = r} u(y) d\sigma_r(y)$$

valid for all $x$ and $r$ such that $B(x, r) \subset \Omega$.

[Hint: Use Green’s formula with $u$ and a Green’s function for the Dirichlet problem on $B(x, r)$.]