1. Let \( \mathbb{H}^n \) be the upper half-space. Given \( m \in \mathbb{N} \), construct an extension operator \( \text{ext} : C^m(\mathbb{H}^n) \to C^m(\mathbb{R}^n) \) such that
\[
(\text{ext} \, u)|_{\mathbb{H}^n} = u, \quad u \in C^m(\mathbb{H}^n).
\]
Using localization arguments this result can be extended to cover extension from a domain with smooth boundary.

2. Let Banach spaces \( E_j, j = 0, 1, 2 \), be given with
\[
E_2 \hookrightarrow E_1 \hookrightarrow E_0.
\]
Show that, given \( \varepsilon > 0 \), there is a constant \( c_\varepsilon > 0 \) such that
\[
\|u\|_{E_1} \leq \varepsilon \|u\|_{E_2} + c_\varepsilon \|u\|_{E_0}, \quad u \in E_2.
\]

3. Show that the trace operator \( \gamma_{\partial \mathbb{H}^n} \) satisfies
\[
\gamma_{\partial \mathbb{H}^n}(H^2(\mathbb{R}^n)) = H^{3/2}(\partial \mathbb{H}^n).
\]

4. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary, \( \alpha > 0 \) and \( f \in L^2(\Omega) \). Find the boundary value problem for which
\[
\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx + \alpha \int_{\partial \Omega} uv \, d\sigma = \int_\Omega fv \, dx \quad \forall v \in H^1(\Omega)
\]
is the appropriate weak formulation and show that it possesses a unique solution.

5. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary and find the natural weak \( L^2(\Omega) \)-formulation of the bvp
\[
\begin{cases}
\Delta^2 u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
\partial_n u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
and prove that it has a unique solution.

Homework due by Wednesday, December 17 2014