1. Let $\mathcal{A}$ be the general elliptic second order differential operator in divergence from on a bounded domain $\Omega$ with smooth boundary, that is,$$
abla \cdot (A \nabla u) + (b \nabla u) + cu$$where the coefficients satisfy the assumptions of Chapter 8 of class. Let $\Phi \in \text{Diff}^2(\Omega, \tilde{\Omega})$, that is, $\Phi$ is invertible and$$\Phi \in C^2(\Omega, \tilde{\Omega}), \quad \Psi := \Phi^{-1} \in C^2(\tilde{\Omega}, \Omega).$$Letting $y := \Phi(x)$ and define $\tilde{u}(y) := u(\Psi(y))$, compute the operator $\tilde{\mathcal{A}}$ in the new variables, that is the operator satisfying$$\tilde{\mathcal{A}}u = \tilde{A}\tilde{u}.$$

2. Prove that the Neumann problem$$\begin{cases}
-\Delta u = f \text{ in } \Omega, \\
\partial_{\nu} u = 0 \text{ on } \partial \Omega
\end{cases}$$on a bounded domain with smooth boundary has a solution if and only if $\int_{\Omega} f \, dx = 0$.

3. Let $H(t, x)$ be the heat kernel (the fundamental solution for the heat equation introduced in class) and show that $T$ defined through$$\begin{cases}
(T(t)u)(x) := \int_{\mathbb{R}^n} H(t, x - y)u(y) \, dy, \quad x \in \mathbb{R}^n \\
T(0)u := u
\end{cases}$$is a $C_0$-semigroup of contractions on $L_2(\mathbb{R}^n)$ but NOT on $L_\infty(\mathbb{R}^n)$. A $C_0$-semigroup $T$ on a Banach space $E$ is called analytic if it allows for an analytic strongly continuous extension to a sector $\Sigma_\delta = \{\text{arg}(z) < \delta\}$ of the complex plane for some $\delta \in (0, \pi/2]$, that is, if

(i) $T(0) = \text{id}_E$, $T(z_1 + z_2) = T(z_1)T(z_2)$, $z_1, z_2 \in \Sigma_\delta$.
(ii) $T : \Sigma_\delta \to \mathcal{L}(E)$ is analytic.
(iii) $\lim_{\delta \to 0} T(z)x = x$ for all $x \in E$.

It can be shown that the above conditions are equivalent to

(i) $T(t)E \subset \text{dom}(A)$, $t > 0$.
(ii) $\|tAT(t)\|_{\mathcal{L}(E)} \leq c < \infty$, $t > 0$.
where $-A : \text{dom}(A) \subset E \rightarrow E$ is the generator of $T$. Show that the $C_0$-semigroup of problem 1 is analytic.
4. Let \( A : \text{dom}(A) \subset E \longrightarrow E \) be defined through
\[
E = L^2(0, 1),
\]
\[
\text{dom}(A) = \left\{ u \in H^2(0,1) \mid u(0) = u(1) = 0 \right\},
\]
\[
Au = -\partial_{xx} u, \ u \in \text{dom}(A),
\]
and show that \(-A\) generates an analytic \( C_0 \)-semigroup on \( E \).

5. For a bounded domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary, for \( b \in L^\infty(\Omega) \) and \( c \in L^\infty(\Omega) \) let \( A \) be the operator induced by the Dirichlet form
\[
a(u,v) = \int_{\Omega} \left[ (\nabla u \cdot \nabla v) + (b \nabla u)v + cuv \right] \, dx, \ u,v \in \overset{\circ}{H}^1(\Omega)
\]
on \( H^{-1}(\Omega) \). Show that it generates a \( C_0 \)-semigroup on \( H^{-1}(\Omega) \).

Homework due by Friday, January 16 2015