Assignment 9

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1. Define $T(t) \in \mathcal{L}\big(\mathrm{L}_2(\mathbb{R}^n)\big)$ through
$$\begin{cases} (1-\triangle)^{-t} = \mathcal{F}^{-1}(1+|\xi|^2)^{-t}\mathcal{F}\,, & t>0 \\ \mathrm{id}_{\mathrm{L}_2(\mathbb{R}^n)}\,, & t=0\,. \end{cases}$$
Show that T is a C_0-semigroup on $\mathrm{L}_2(\mathbb{R}^n)$. What is its generator?

2. Let $A \in \mathcal{G}(E)$, $x \in E$ and $f \in \mathrm{C}^{1-}\big([0,\infty)\times E, E\big)$ and prove that
$$\begin{cases} \dot{u} + Au = f(t,u)\,,\ t>0 \\ \quad u(0) = x \end{cases}$$
has a unique local mild solution $u(\cdot,x) \in \mathrm{C}\big([0,t_x^+),E\big)$ for some $t_x^+ > 0$.
[Hint: Use Banach Fixed-Point Theorem combined with the Variation-of-Constants-Formula]

3. Let $-A : \mathrm{dom}(A) \subset E \longrightarrow E$ be the generator of an analytic C_0-semigroup T on E. Let $f \in \mathrm{C}^\rho\big([0,T],E\big)$ for some $\rho \in (0,1)$ and show that the mild solution $u : [0,T] \to E$ of
$$\dot{u} + Au = f(t)\,,\ u(0) = x \in E\,,$$
given by
$$u(t) = T(t)x + \int_0^t T(t-\tau)f(\tau)\,d\tau\,,\ t \in [0,T]$$
is actually differentiable for $t > 0$.

4. Let $A \in \mathbb{C}^{n\times n}$ and show that
$$e^{-tA} = \frac{1}{2\pi i}\int_{\partial\mathbb{B}(0,R)} e^{\lambda t}(\lambda + A)^{-1}\,d\lambda\,,$$
where $R > 0$ is such that $\sigma(-A) \subset \mathbb{B}(0,R)$ and the integration is counterclockwise.
[Hint: Show that both sides of the equation satisfy the same ODE and prove that they have the same initial value by reducing the problem to the case where A is a Jordan block]

5. Let $f \in \mathrm{C}\big(\mathbb{B}(0,1),\mathbb{R}^n\big)$ be such that
$$|f(x)| \le 1 \text{ for } |x| = 1\,.$$
Show that f has a fixed point in $\mathbb{B}(0,1)$.

The Homework is due Friday, January 30 2015