1. It can be shown that any distribution can be viewed as some derivative of a continuous function. More precisely
\[ \forall u \in \mathcal{D}'(\Omega) \forall K = \bar{K} \subset \Omega \exists f \in C(\Omega) \text{ and } \alpha \in \mathbb{N}^n \text{ s.t.} \]
\[ \langle u, \varphi \rangle = (-1)^{|\alpha|} \int_{\Omega} f(x) \partial^{\alpha} \varphi(x) \, dx \forall \varphi \in \mathcal{D}_K(\Omega). \]
For a proof see the book *Functional Analysis* by W. Rudin (p. 167).

Let now \( \delta \in \mathcal{D}'(\mathbb{R}^2) \) and find a simple representation of it as some derivative of a continuous function.

2. For \( u \in \mathcal{D}'(\mathbb{R}^n) \) and \( v \in \mathbb{R}^n \) define \( \tau_v u \in \mathcal{D}'(\mathbb{R}^n) \) through
\[ \langle \tau_v u, \varphi \rangle = \langle u, \tau_{-v} \varphi \rangle \forall \varphi \in \mathcal{D}(\mathbb{R}^n) \]
where \( \tau_{v} \varphi = \varphi(\cdot - v) \). Show that \( \tau_v u \) is well-defined and prove that
\[ \frac{\tau_{-h e_j} u - u}{h} \to \partial_j u \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ as } h \to 0 \]
for \( j = 1, \ldots, n \).

3. Let \( \varphi, \psi \in \mathcal{D}(\Omega) \) and \( \alpha \in \mathbb{N}^n \). Determine the real numbers \( c_{\alpha \beta}, \beta \leq \alpha \) for which the formula
\[ \partial^{\alpha} (\varphi \psi) = \sum_{\beta \leq \alpha} c_{\alpha \beta} \partial^{\alpha-\beta} \varphi \partial^{\beta} \psi \]
holds. By \( \beta \leq \alpha \) it is meant that \( \beta_j \leq \alpha_j \) for \( j = 1, \ldots, n \).

4. Show that \( u \in \mathcal{D}'((0, \infty)) \) given through
\[ \langle u, \varphi \rangle := \sum_{m=1}^{\infty} \left( \partial^m \varphi \right)(\frac{1}{m}), \varphi \in \mathcal{D}((0, \infty)) \]
is well-defined. Show that \( u \) has infinite order and that it cannot be extended to a distribution of the real line.

5. Let \( \Omega \subset \mathbb{R}^n \) be an open and bounded domain with smooth boundary \( \partial \Omega \) and denote by \( \chi_{\Omega} \) its characteristic function. Compute \( \nabla \chi_{\Omega} \) in the sense of distributions and determine its support.

Homework due by Friday, October 30 2015