Assignment 3

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1. Let $C^\infty[-1,1] \ni a \geq a_0 > 0$ and $y \in [-1,1]$. Compute $G(\cdot,y)$ satisfying
$$-\partial_x\big(a(x)\partial_x G(\cdot,y)\big) = \delta_y$$
and $G(-1,y) = G(1,y) = 0$. Let $f \in L^1(-1,1)$ and show that u defined through
$$u(x) = \int_{-1}^{1} G(x,y)f(y)\,dy\,,\ x \in (-1,1)$$
solves (in which sense?) the following boundary value problem
$$\begin{cases} -\partial_x\big(a(x)\partial_x u\big) = f(x)\,,\ x \in (-1,1)\,, \\ u(-1) = u(1) = 0\,. \end{cases}$$

2. Let $\omega \in \mathbb{R}$ and find the solution of
$$(\partial_x^2 + \omega^2)G = \delta \text{ in } \mathcal{D}'(\mathbb{R})$$
[Hint: Let $G = Hf$ for $f \in C^\infty(\mathbb{R})$ and the Heaviside function H]

3. Let $T \in \mathcal{E}'(\mathbb{R}^n)$ with $\mathrm{supp}(T) = K$ be of finite order $m \in \mathbb{N}$. Show that
$$\langle T,\phi\rangle = 0 \text{ for any } \phi \in \mathcal{E}(\mathbb{R}^n) \text{ with } \partial^\alpha\phi\big|_K \equiv 0 \text{ for } |\alpha| \leq m\,.$$

4. Let E be a given vector space. A function $p : E \to [0,\infty)$ is called *seminorm* if
$$p(x+y) \leq p(x) + p(y) \text{ and } p(\alpha x) = |\alpha|\,p(x)$$
for $x,y \in E$ and $\alpha \in \mathbb{K}$. A family of seminorms $\{p_\lambda : \lambda \in \Lambda\}$ is called *separating* if
$$p_\lambda(x) = 0 \text{ for each } \lambda \in \Lambda \text{ implies } x = 0\,.$$
Now, a set $X \subset E$ is called *open* if for each $x \in X$ there are finitely many $\lambda_j \in \Lambda$ and $\epsilon_j > 0$ such that
$$x + \cap_{j=1,\dots,m}\, p_{\lambda_j}^{-1}\big([0,\epsilon_j)\big) \subset X\,.$$
Show that the collection of all open sets is a topology on E. E endowed with this topology is called *locally convex space*. A linear map
$$u : E \to E$$
is then continuous iff preimages of open sets are open. Find a characterization of continuity by means of inequalities for the seminorms

p_λ. Prove that the space $\mathcal{E}(\Omega)$ defined in class is a locally convex space with seminorms

$$\{p_{K,m} : K = \bar{K} \subset\subset \Omega\,,\ m \in \mathbb{N}\}$$

defined through $p_{K,m}(\varphi) = \sup_{x\in K,|\alpha|\leq m} \left|D^\alpha \varphi(x)\right|$.

5. Let $\psi \in \mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ with $\psi = 1$ on $\mathbb{B}(0,1)$ and define ψ_k through

$$\psi_k(x) = \psi(\frac{x}{k})\,,\ x \in \mathbb{R}^n\,.$$

Show that $\psi_k \to \mathbf{1}$ $(k \to \infty)$ in $\mathcal{E} = \mathcal{E}(\mathbb{R}^n)$. Then prove that $\mathcal{D}$ is dense in $\mathcal{E}$ as well as $\mathcal{E}'$ in $\mathcal{D}'$.

The Homework is due by Monday, November 9 2015