Assignment 7

1. Let $\mathbb{H}^n$ be the upper half-space. Given $m \in \mathbb{N}$ construct an extension operator $\operatorname{ext} : \mathrm{C}^m(\overline{\mathbb{H}^n}) \to \mathrm{C}^m(\mathbb{R}^n)$ such that
$$(\operatorname{ext} u)\big|_{\mathbb{H}^n} = u\,,\ u \in \mathrm{C}^m(\overline{\mathbb{H}^n})\,.$$
Using localization arguments this result can be extended to cover extension from a domain with smooth boundary.

2. Let Banach spaces E_j, $j = 0, 1, 2$, be given with
$$E_2 \overset{d}{\hookrightarrow} E_1 \hookrightarrow E_0\,.$$
Show that, given $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that
$$\|u\|_{E_1} \le \varepsilon \|u\|_{E_2} + c_\varepsilon \|u\|_{E_0}\,,\ u \in E_2\,.$$

3. Show that the trace operator $\gamma_{\partial\mathbb{H}^n}$ satisfies
$$\gamma_{\partial\mathbb{H}^n}\big(\mathrm{H}^2(\mathbb{R}^n)\big) = \mathrm{H}^{3/2}(\partial\mathbb{H}^n)\,.$$

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $\alpha > 0$ and $f \in \mathrm{L}_2(\Omega)$. Find the boundary value problem for which
$$\int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega uv\, dx + \alpha \int_{\partial\Omega} uv\, d\sigma_{\partial\Omega} = \int_\Omega fv\, dx\ \forall v \in \mathrm{H}^1(\Omega)$$
is the appropriate weak formulation and show that it possesses a unique solution.

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and find the natural weak $\mathrm{L}_2(\Omega)$-formulation of the bvp
$$\begin{cases} \triangle^2 u = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \\ \partial_\nu u = 0 & \text{on } \partial\Omega\,, \end{cases}$$
and prove that it has a unique solution.

Homework due by Friday, December 11 2015