Assignment 1

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1. Show that $v.p.\frac{1}{x}$ is a well-defined distribution. Recall that
$$\langle v.p.\frac{1}{x}, \varphi\rangle = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,dx$$
for any test function $\varphi \in \mathcal{D}(\mathbb{R})$. Then prove that
$$u_\varepsilon^{\pm} \to \mp i\pi\delta + v.p.\frac{1}{x} \text{ as } \varepsilon \downarrow 0$$
for $u_\varepsilon^{\pm}(x) := \frac{1}{x\pm i\varepsilon}$, $x \in \mathbb{R}$, in the sense of distributions.

2. Compute f' and f'' for $f(x) = \log|x|$, $x \in \mathbb{R}$ and $f(x) = |x|$, $x \in \mathbb{R}$, respectively, in the sense of distributions.

3. Let $\rho : \mathbb{R}^n \mapsto \mathbb{R}$ be an integrable function with
$$\mathrm{supp}(\rho) \subset \mathbb{B}(0,1) \text{ and } \int_{\mathbb{R}^n}\rho(x)\,dx = 1\,.$$
Show that $\rho_\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$ for
$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})\,,\ x \in \mathbb{R}^n\,.$$

4. Consider the function u defined as
$$u(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}\,,\ (x,y) \in \mathbb{R}\times(0,\infty)$$
Then $u \in \mathrm{C}^\infty(\mathbb{R}\times(0,\infty))$. Compute $\lim_{y\to 0} u(\cdot, y)$.

5. Given a continuous function $f \in \mathrm{C}(\Omega)$ with $\mathrm{supp}(f) \subset\subset \Omega$, show that a sequence $(\varphi_j)_{j\in\mathbb{N}}$ of test functions in $\mathcal{D}(\Omega)$ can be found such that
$$\|\varphi_j - f\|_\infty := \sup_{x\in\Omega}|\varphi_j(x) - f(x)| \longrightarrow 0 \text{ as } j \to \infty\,.$$

Homework due by Monday, October 9 2017