Math 295 FALL TERM 2017

Assignment 4

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1. Compute the Fourier transform with respect to the variable x of the following functions:
   (i) $u_y(x) = e^{-y|x|}\,,\ y > 0\,,\ x \in \mathbb{R}$.
   (ii) $u(x) = e^{-\frac{|x|^2}{2}}\,,\ x \in \mathbb{R}^n\,.$
   (iii) $u(x) = \frac{y^2 - x^2}{(y^2+x^2)^2}\,,\ y > 0\,,\ x \in \mathbb{R}$.

2. Show that G defined through $G(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ for $x \in \mathbb{R}$ and $y > 0$ is harmonic, that is, $\triangle G = 0$, and conclude that

$$u_g(x,y) := \int_{-\infty}^{\infty} G(x - \tilde{x}, y) g(\tilde{x})\, d\tilde{x}\,,\ (x,y) \in \mathbb{R} \times (0,\infty)$$

   represents a solution of

$$\begin{cases} \triangle u &= 0 \text{ in } \mathbb{R} \times (0,\infty) \\ u &= g \text{ on } \mathbb{R} \times \{0\} \end{cases}$$

   for $g \in \mathrm{L}_1(\mathbb{R})$. What is $\lim_{y\to\infty} u_g(\cdot, y)$?

3. Let $f \in \mathcal{D}(\mathbb{R}^n)$ with $\mathrm{supp}(f) \subset \mathbb{B}(0,R)$ for $0 < R < \infty$. Show that its Fourier transform $\hat{f}$ is holomorphic and satisfies

$$|\hat{f}(\xi + i\eta)| \le c_N \frac{1}{(1+|\xi|^2)^{N/2}} e^{R|\eta|}\,,\ (\xi,\eta) \in \mathbb{R}^{2n}$$

   for any $N \in \mathbb{N}$ and some constant c_N.

4. Assume $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ and let

$$T : \mathbb{R} \to \mathcal{S}(\mathbb{R}^n)\,,\ t \to \varphi(\cdot - t\,a)\,.$$

   Prove that $T \in \mathrm{C}^1\big(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)\big)$ and compute

$$\dot{T}(0) \in \mathcal{L}\big(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)\big) \hat{=} \mathcal{S}(\mathbb{R}^n)\,.$$

5. Let $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and consider the homogeneous heat equation

$$\begin{cases} u_t - \triangle u = 0\,, & \text{in } (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0\,, & \text{in } \mathbb{R}^n \end{cases}$$

   Prove that it has a unique solution

$$u \in \mathrm{C}^\infty\big([0,\infty), \mathcal{S}(\mathbb{R}^n)\big)$$

   and derive a representation formula for it.

Homework due by Friday, November 17 2017