Assignment 6

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and prove $Poincar\acute{e}$'s Inequality

$$||u||_{\mathcal{L}_p(\Omega)} \le c||\nabla u||_{\mathcal{L}_p(\Omega)}, \ u \in \overset{\circ}{\mathcal{W}}_p^1(\Omega).$$

What does the constant c depend on? Is the boundedness assumption really necessary?

2. Let $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty)$ and define

$$W_p^m(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) \mid \partial^{\alpha} u \in L_p(\Omega), |\alpha| \le m \right\}.$$

Show that $\mathbf{W}_p^m(\Omega)$ is a Banach space if endowed with the norm $\|\cdot\|_{m,p}$ defined by

$$||u||_{m,p} = \left(\sum_{|\alpha| \le m} ||\partial^{\alpha} u||_{\mathbf{L}_p(\Omega)}^p\right)^{1/p}, \ u \in \mathbf{W}_p^m(\Omega).$$

Prove that $W_p^1(0,1) \hookrightarrow BUC^{1-1/p}([0,1])$.

[Hint: Use the fact that $C^1([0,1])$ is dense in $W^1_p(0,1)$]

- 3. Prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$.
- 4. Let $\Omega = \mathbb{B}(0,1/2)$ and the function u be defined through

$$u(x,y) = \log(\log(\frac{2}{\sqrt{x^2 + y^2}})), (x,y) \in \Omega.$$

Then u is obviously not continuous in (x,y)=(0,0). Prove that, however, $u\in H^1(\Omega)$. Let now

$$u(x,y) = xy \lceil \log |\log |(x,y)| - \log \log 2 \rceil, (x,y) \in \Omega.$$

Show that

$$u \in C^1(\bar{\Omega})$$
 and $\partial_i^2 u \in C(\bar{\Omega})$, $j = 1, 2$

is a solution of the Dirichlet problem in the ball with continuous datum but $u \notin C^2(\bar{\Omega})$.

5. Let E be a Banach space and A: $dom(A) \subset E \longrightarrow E$ a linear, possibly unbounded, operator on E. A is said to be invertible if there exists a bounded operator $B \in \mathcal{L}(E)$ such that

$$AB = id_E$$
 and $BA = id_{dom(A)}$.

Such an operator A can fail to be invertible either because it has non trivial kernel (not injective)

$$\ker(A) \neq \{0\}$$

or because it is not surjective

$$\overline{R(A)} \neq E$$

but, also, because its "inverse" is unbounded. Let

$$E = l_2(\mathbb{N}) := \left\{ (x_j)_{j \in \mathbb{N}} \, | \, x_j \in \mathbb{R} \, \forall j \in \mathbb{N} \text{ and } \sum_{j=1}^{\infty} x_j^2 < \infty \right\}$$

with the norm naturally induced by the scalar product

$$(x|y) = \sum_{j=1}^{\infty} x_j y_j, \ x, y \in l_2(\mathbb{N}).$$

For each one of the ways described find an operator A on $l_2(\mathbb{N})$ which fails to be invertible in that and no other way. In general the set

$$\sigma(A) = \{ \lambda \in \mathbb{C} \mid \lambda - A \text{ is not invertible} \} \subset \mathbb{C}$$

is called spectrum of A. Show that it is a closed set.