

## Assignment 10

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1. Let  $0 \neq \alpha \in \mathbb{R}$  and consider the initial value problem for Euler's equation

$$\sum_{k=1}^n x_k \partial_{x_k} u = \alpha u, \quad u(x_1, \dots, x_{n-1}, 1) = g(x_1, \dots, x_{n-1}).$$

Show that its solution satisfies the functional equation

$$u(\lambda x) = \lambda^\alpha u(x), \quad x \neq 0, \lambda > 0.$$

What is the behavior of the solution at  $x = 0$ ?

2. Consider the quasilinear initial value problem

$$\begin{cases} u_t + u u_x = 0, & (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, x) = g(x), & x \in \mathbb{R} \end{cases}$$

Solve the equation and analyze the possible onset of singularities. What conditions on  $g$  would prevent the solution from developing singularities?

3. Let  $u \in C^1(\mathbb{B}(0, 1))$  be a solution of

$$a(x, y)u_x + b(x, y)u_y = -u$$

and assume that  $a(x, y)x + b(x, y)y > 0$  for  $(x, y) \in \mathbb{S}^1$ . Show that  $u \equiv 0$  then.

4. Consider the equation

$$\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0.$$

Any function  $u$  with

$$\int_{\mathbb{R}^2} [R(u)\phi_y + S(u)\phi_x] d(x, y) = 0, \quad \phi \in C_0^\infty(\mathbb{R}^2)$$

is called *weak solution*. Assume that  $u$  is continuously differentiable away from some curve parametrized by  $(s(y), y)$ ,  $y \in \mathbb{R}$  across which it has a jump discontinuity. Conclude that

$$s'(y) = \frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)}$$

where  $u^\pm$  indicate the one sided limits of  $u$  approaching the curve.

5. Consider the eikonal equation

$$c^2(u_x^2 + u_y^2) = 1$$

Let  $\gamma_t$  be the level line  $[u(x, y) = t]$  of a solution  $u$ . Show that a point  $(x, y)$  moves in a direction perpendicular to  $\gamma_t$  at constant speed  $c$  (along a characteristic).