Assignment 12

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1. Assume that $f \in C([0,1])$ and that

   $$F \in C([0,1] \times [0,1] \times \mathbb{R}) \,, \quad \partial_u F \in C([0,1] \times [0,1] \times \mathbb{R})$$

   and consider the integral equation

   $$u(x) = \int_0^1 F(x, y, u(y)) \, dy + f(x) \,, \quad 0 \le x \le 1 \,.$$

   Show that, if $\|\partial_u F\|_\infty < 1$, the integral equation has a unique solution $u \in C([0,1])$.

2. Let $f \in C(\mathbb{B}(0,1), \mathbb{R}^n)$ be such that
   $$|f(x)| \le 1 \text{ for } |x| = 1 \,.$$
   Show that f has a fixed point in $\mathbb{B}(0,1)$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. Prove that the equation
   $$\begin{cases} -\triangle u = e^{-u} & \text{in } \Omega \,, \\ \quad\;\; u = 0 & \text{on } \partial\Omega \,, \end{cases}$$
   possesses a solution.

4. (*Kolmogoroff*) Show that a subset $K \subset L_p(\mathbb{R}^n)$ ($1 \le p < \infty$) is compact iff
   (i) K is closed and bounded.
   (ii) $\int_{|x| \ge N} |f(x)|^p \, dx \to 0$, $N \to \infty$, uniformly in $f \in K$.
   (iii) $\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, dx \to 0$, $|h| \to 0$, uniformly in $f \in K$.
   [Hint: Use the density of test functions in $L_p(\mathbb{R}^n)$, the strong continuity of the translation semigroup on $L_p(\mathbb{R}^n)$ and Arzéla-Ascoli.]

5. Let $1 \le p < \infty$ and prove that
   $$\Big(\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p \, dx \Big)^{1/p} \le |h| \, \|u\|_{1,p}$$
   for $u \in W_p^1(\mathbb{R}^n)$. Use this estimate and Kolmogoroff's characterization of compactness to show that $W_p^1(\Omega) \hookrightarrow L_p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open and bounded.

<div align="center">Homework due by Thursday, April 24 2008.</div>