

Assignment 13

Let $\Omega \subset \mathbb{R}^n$ be open. A map $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is called *Carathéodory function* whenever

- (i) $f(\cdot, s) : \Omega \rightarrow \mathbb{R}$ is measurable for every $s \in \mathbb{R}$.
- (ii) $f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous for almost every $x \in \Omega$.

1. Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function and $p, q \geq 1$. Assume that

$$|f(x, s)| \leq c|s|^{p/q} + g(x)$$

for some $g \in L_q(\Omega)$. Prove that the Nemytzki operator (substitution operator) $N_f : L_p(\Omega) \rightarrow L_q(\Omega)$ defined through

$$(N_f u)(x) := f(x, u(x)), \quad x \in \Omega$$

is well-defined, continuous and maps bounded sets onto bounded sets.

2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Assume that the Carathéodory function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$\underline{f} \leq \frac{f(x, u) - f(x, v)}{u - v} \leq \bar{f} \quad \text{and} \quad f(\cdot, 0) \in L_2(\Omega)$$

with $\sigma(-\Delta_D) \cap [\underline{f}, \bar{f}] = \emptyset$. Show that

$$\begin{cases} \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

possesses a unique weak solution $u \in \mathring{H}^1(\Omega)$.

3. Let H be a Hilbert space. Prove that

$$x_n \rightarrow x, y_n \rightarrow y \quad (n \rightarrow \infty) \Rightarrow (x_n | y_n) \rightarrow (x | y) \quad (n \rightarrow \infty).$$

4. Let E be a normed vector space and $A \in \mathcal{L}(E)$ a compact operator. Show that

$$\dim(\ker[(\lambda - A)^n]) < \infty \quad \text{for any } \lambda \neq 0 \text{ and } n = 1, 2, \dots$$

[Use the fact that $\overline{\mathbb{B}}_E(0, 1)$ is compact iff $\dim(E) < \infty$ and that eigenvectors to different eigenvalues are linearly independent.]

5. Let $K \in \mathcal{L}(H)$ be a compact operator. Show that it maps weakly convergent sequences to convergent sequences.

Homework due on Wednesday, May 7 2008