Assignment 14

1. (Pohožaev's identity) Assume that $g \in \mathrm{C}(\mathbb{R},\mathbb{R})$, $G(u) = \int_0^u g(v)\,dv$ and that $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary. Let u be a classical solution of
$$\begin{cases} -\triangle u = g(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$
and show that it satisfies
$$n\int_\Omega G(u)\,dx + \frac{2-n}{2}\int_\Omega u\,g(u)\,dx = \frac{1}{2}\int_{\partial\Omega} (\nabla u\cdot\nu)^2(x\cdot\nu)\,d\sigma$$
[Hint: Use Gauss theorem with the vector field $V(x) = (x\cdot\nabla u)\nabla u$.]

2. Use Pohožaev's identity to prove that no nontrivial solution can exist for
$$\begin{cases} -\triangle u = |u|^p & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$
if $p > \frac{n+2}{n-2}$ and Ω is a star-shaped bounded Lipschitz domain in $\mathbb{R}^n$.

3. Let X be a normed vector space. Prove that a convex functional $\phi : X \to \mathbb{R}$ is continuous at $x \in X$ if it is bounded in a neighborhood of x. Give an example of a convex functional which is nowhere continuous.

4. Let $\beta \in \mathrm{C}^\infty(\mathbb{R})$ satisfying $\beta'(\mathbb{R}) \subset [\delta,\sigma]$ for δ , $\sigma \in (0,\infty)$. Give a weak formulation of
$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ \partial_\nu u + \beta(u) = 0 & \text{on } \Omega\,. \end{cases}$$
in an open bounded domain $\Omega \subset \mathbb{R}^n$ and prove that it possesses a weak solution.

5. For an open and bounded $\Omega \subset \mathbb{R}^n$ let
$$\mathcal{A} = \big\{u \in \mathrm{H}_0^1(\Omega,\mathbb{R}^m)\,\big|\, u = g \text{ on } \partial\Omega\,,\ |u| = 1 \text{ a.e.}\big\}\,.$$
Show that ϕ defined by $\phi(u) = \frac{1}{2}\int_\Omega |Du(x)|^2\,dx$ has at least one minimizer in $\mathcal{A}$ (if $\mathcal{A} \neq \emptyset$) and that any minimizer satisfies
$$\int_\Omega Du(x) : Dv(x)\,dx = \int_\Omega |Du(x)|^2 u(x)v(x)\,dx\,,$$
$$v \in \mathrm{H}_0^1(\Omega,\mathbb{R}^m) \cap \mathrm{L}_\infty(\Omega,\mathbb{R}^m)\,.$$

Homework due by Wednesday, May 21 2008