Assignment 15

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1. Consider the following *porous medium equation*:
$$u_t - \triangle(u^\gamma) = 0 \text{ in } \mathbb{R}^n \times (0,\infty)\,.$$
(i) Show that there exists a solution of the form $u(t,x) = v(t)w(x)$.
(ii) Find a solution of the form $u(t,x) = \frac{1}{t^\alpha} v\big(\frac{x}{t^\beta}\big)$.
What are the main features of these solutions?

2. Find a transformation $w = \Phi(u)$ which reduces the nonlinear equation
$$\begin{cases} u_t - a\triangle u + b|\nabla u|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty)\,, \\ u = g & \text{in } \mathbb{R}^n \times \{0\} \end{cases}$$
to a linear one and use the latter to find a solution of the former.

3. Use the solution to the previous problem to find a solution for the *viscous Burger's equation*
$$\begin{cases} u_t - au_{xx} + u\,u_x = 0 & \text{in } \mathbb{R} \times (0,\infty)\,, \\ u = g & \text{in } \mathbb{R} \times \{0\}\,. \end{cases}$$
[Hint: Derive an equation for $w(t,x) = \int_{-\infty}^x u(t,y)\,dy$.]

4. Consider *Euler's equation*
$$\begin{cases} u_t - u \cdot Du = -\nabla p & \text{in } \mathbb{R}^3 \times (0,\infty)\,, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty)\,, \\ u = g & \text{in } \mathbb{R}^3 \times \{0\}\,. \end{cases}$$
Assume that there exists v such that $u = \nabla v$ and derive a simpler equation for v. Once v is obtained, how can p be derived?

5. Consider the *minimal surface equation*
$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla|^2}}\Big) = 0 \text{ in } \mathbb{R}^2$$
Let $|D^2u| \neq 0$ at $x_0 \in \mathbb{R}^2$ and derive the equation satisfied by
$$v(p) := x(p)\cdot p - u\big(x(p)\big)$$
in a neighborhood of x_0, where $x(p)$ is the unique solution of
$$p = \nabla u(x)\,.$$

Homework due Friday, June 6 2008