

Assignment 3

1. Let $C^\infty[-1, 1] \ni a \geq a_0 > 0$ and $y \in [-1, 1]$. Compute $G(\cdot, y)$ satisfying

$$-\partial_x(a(x)\partial_x G(\cdot, y)) = \delta_y$$

and $G(-1, y) = G(1, y) = 0$. Let $f \in L^1(-1, 1)$ and show that u defined through

$$u(x) = \int_{-1}^1 G(x, y)f(y) dy, \quad x \in (-1, 1)$$

solves (in which sense?) the following boundary value problem

$$\begin{cases} -\partial_x(a(x)\partial_x u) = f(x), & x \in (-1, 1), \\ u(-1) = u(1) = 0. \end{cases}$$

2. Let $\omega \in \mathbb{R}$ and find the solution of

$$(\partial_x^2 + \omega^2)G = \delta \text{ in } \mathcal{D}'(\mathbb{R})$$

[Hint: Let $G = Hf$ for $f \in C^\infty(\mathbb{R})$ and the Heaviside function H]

3. Let $T \in \mathcal{E}'(\mathbb{R}^n)$ with $\text{supp}(T) = K$ be of finite order $m \in \mathbb{N}$. Show that

$$\langle T, \phi \rangle = 0 \text{ for any } \phi \in \mathcal{E}(\mathbb{R}^n) \text{ with } \partial^\alpha \phi|_K \equiv 0 \text{ for } |\alpha| \leq m$$

4. Let E be a given vector space. A function $p : E \rightarrow [0, \infty)$ is called *seminorm* if

$$p(x+y) \leq p(x) + p(y) \text{ and } p(\alpha x) = |\alpha| p(x)$$

for $x, y \in E$ and $\alpha \in \mathbb{K}$. A family of seminorms $\{p_\lambda : \lambda \in \Lambda\}$ is called *separating* if

$$p_\lambda(x) = 0 \text{ for each } \lambda \in \Lambda \text{ implies } x = 0.$$

Now, a set $X \subset E$ is called *open* if for each $x \in X$ there are finitely many $\lambda_j \in \Lambda$ and $\epsilon_j > 0$ such that

$$x + \bigcap_{j=1, \dots, m} p_{\lambda_j}^{-1}((0, \epsilon_j)) \subset X.$$

Show that the collection of all open sets is a topology on E . E endowed with this topology is called *locally convex space*. A linear map

$$u : E \rightarrow E$$

is then continuous iff preimages of open sets are open. Find a characterization of its continuity by means of inequalities for the seminorms

p_λ . Prove that the space $\mathcal{E}(\Omega)$ defined in class is a locally convex space with seminorms

$$\{p_{K,m} : K = \bar{K} \subset\subset \Omega, m \in \mathbb{N}\}$$

defined through $p_{K,m}(\varphi) = \sup_{x \in K, |\alpha| \leq m} |D^\alpha \varphi(x)|$.

5. Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi = 1$ on $\mathbb{B}(0, 1)$ and define ψ_k through

$$\psi_k(x) = \psi\left(\frac{x}{k}\right), x \in \mathbb{R}^n.$$

Show that $\psi_k \rightarrow \mathbf{1}$ ($k \rightarrow \infty$) in $\mathcal{E}(\mathbb{R}^n)$. Then prove that

$\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{E}(\mathbb{R}^n)$ as well as $\mathcal{E}(\mathbb{R}^n)'$ in $\mathcal{D}(\mathbb{R}^n)'$.