

## Assignment 4

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1. Compute the Fourier transform with respect to the variable  $x$  of the following functions:

(i)  $u_y(x) = e^{-y|x|}$ ,  $y > 0$ ,  $x \in \mathbb{R}$ .

(ii)  $u(x) = e^{-\frac{|x|^2}{2}}$ ,  $x \in \mathbb{R}^n$ .

(iii)  $u(x) = \frac{y^2 - x^2}{(y^2 + x^2)^2}$ ,  $y > 0$ ,  $x \in \mathbb{R}$ .

2. Show that  $G$  defined through  $G(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$  for  $x \in \mathbb{R}$  and  $y > 0$  is harmonic, that is,  $\Delta G = 0$ , and conclude that

$$u_g(x, y) := \int_{-\infty}^{\infty} G(x - \tilde{x}, y) g(\tilde{x}) d\tilde{x}, \quad (x, y) \in \mathbb{R} \times (0, \infty)$$

represents a solution of

$$\begin{cases} \Delta u &= 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u &= g \text{ on } \mathbb{R} \times \{0\} \end{cases}$$

for  $g \in L_1(\mathbb{R})$ . What is  $\lim_{y \rightarrow \infty} u_g(\cdot, y)$ ?

3. Let  $f \in \mathcal{S}(\mathbb{R}^n)$  with  $\text{supp}(f) \subset \mathbb{B}(0, R)$  for  $0 < R < \infty$ . Show that its Fourier transform  $\hat{f}$  is holomorphic and satisfies

$$|\hat{f}(\xi + i\eta)| \leq c_N \frac{1}{(1 + |\xi|^2)^{N/2}} e^{R|\eta|}, \quad (\xi, \eta) \in \mathbb{R}^{2n}.$$

4. Assume  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ ,  $a \in \mathbb{R}^n$  and let

$$T : \mathbb{R} \rightarrow \mathcal{S}(\mathbb{R}^n), \quad t \rightarrow \varphi(\cdot - ta).$$

Prove that  $T \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$  and compute

$$\dot{T}(0) \in \mathcal{L}(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)) \hat{=} \mathcal{S}(\mathbb{R}^n).$$

5. Let  $u_0 \in \mathcal{S}(\mathbb{R}^n)$  and consider the homogeneous heat equation

$$\begin{cases} u_t - \Delta u = 0, & \text{in } (0, \infty) \times \mathbb{R}^n \\ u(0) = u_0, & \text{in } \mathbb{R}^n \end{cases}$$

Prove that it has a unique solution

$$u \in C^\infty([0, \infty), \mathcal{S}(\mathbb{R}^n))$$

and derive a representation formula for it.