

Assignment 6

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and prove *Poincaré's Inequality*

$$\|u\|_{L_p(\Omega)} \leq c \|\nabla u\|_{L_p(\Omega)}, \quad u \in \mathring{W}_p^1(\Omega).$$

What does the constant c depend on? Is the boundedness assumption really necessary?

2. Let $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty)$ and define

$$W_p^m(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid \partial^\alpha u \in L_p(\Omega), |\alpha| \leq m\}.$$

Show that $W_p^m(\Omega)$ is a Banach space if endowed with the norm $\|\cdot\|_{m,p}$ defined by

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p}, \quad u \in W_p^m(\Omega).$$

Prove that $W_p^1(0,1) \hookrightarrow \text{BUC}^{1-1/p}([0,1])$.

[Hint: Use the fact that $C^1([0,1])$ is dense in $W_p^1(0,1)$]

3. Prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$.
4. Let $\Omega = \mathbb{B}(0, 1/2)$ and the function u be defined through

$$u(x, y) = \log\left(\log\left(\frac{2}{\sqrt{x^2 + y^2}}\right)\right), \quad (x, y) \in \Omega.$$

Then u is obviously not continuous in $(x, y) = (0, 0)$. Prove that, however, $u \in H^1(\Omega)$. Let now

$$u(x, y) = xy [\log |\log |(x, y)|| - \log \log 2], \quad (x, y) \in \Omega.$$

Then

$$u \in C^1(\bar{\Omega}) \text{ and } \partial_j^2 u \in C(\bar{\Omega}), \quad j = 1, 2$$

or $u \notin C^2(\bar{\Omega})$, that is, u is a solution of the Dirichlet problem in Ω for a continuous datum but is not twice continuously differentiable.

5. Let E be a Banach space and $A : \text{dom}(A) \subset E \rightarrow E$ a linear, possibly unbounded, operator on E . A is said to be invertible if there exists a bounded operator $B \in \mathcal{L}(E)$ such that

$$AB = \text{id}_E \text{ and } BA = \text{id}_{\text{dom}(A)}.$$

Such an operator A can fail to be invertible either because it has non trivial kernel (not injective)

$$\ker(A) \neq \{0\}$$

or because it is not surjective

$$\overline{R(A)} \neq E$$

but, also, because its “inverse” is unbounded.

Let

$$E = l_2(\mathbb{N}) := \left\{ (x_j)_{j \in \mathbb{N}} \mid x_j \in \mathbb{R} \forall j \in \mathbb{N} \text{ and } \sum_{j=1}^{\infty} x_j^2 < \infty \right\}$$

with the norm naturally induced by the scalar product

$$(x|y) = \sum_{j=1}^{\infty} x_j y_j, \quad x, y \in l_2(\mathbb{N}).$$

For each one of the ways described find an operator A on $l_2(\mathbb{N})$ which fails to be invertible in that way. In general the set

$$\sigma(A) = \{ \lambda \in \mathbb{C} \mid \lambda - A \text{ is not invertible} \} \subset \mathbb{C}$$

is called spectrum of A . Show that it is a closed set.

Homework due by Wednesday, December 5 2007