1. Let $A$ be the general elliptic second order differential operator in divergence from on a bounded domain $\Omega$ with smooth boundary, that is,

$$\mathcal{A}u = -\nabla \cdot (A\nabla u) + (b|\nabla u) + cu$$

where the coefficients satisfy the assumptions of Chapter 8 of class. Let $\Phi \in \text{Diff}^2(\Omega, \tilde{\Omega})$, that is, $\Phi$ is invertible and

$$\Phi \in C^2(\Omega, \tilde{\Omega}), \quad \Psi := \Phi^{-1} \in C^2(\tilde{\Omega}, \Omega).$$

Letting $y := \Phi(x)$ and define $\tilde{u}(y) := u(\Psi(y))$, compute the operator $\tilde{A}$ in the new variables, that is the operator satisfying $\tilde{A}u = \tilde{A}\tilde{u}$.

2. Prove that the Neumann problem

$$\begin{cases}
-\Delta u &= f \text{ in } \Omega, \\
\partial_\nu u &= 0 \text{ on } \partial \Omega
\end{cases}$$

on a bounded domain with smooth boundary has a solution if and only if $\int_{\Omega} f \, dx = 0$.

3. Let $\mathbb{H}^n$ be the upper half-space. Given $m \in \mathbb{N}$ construct an extension operator $\text{ext} : C^m(\mathbb{H}^n) \to C^m(\mathbb{R}^n)$ such that

$$(\text{ext } u)|_{\mathbb{H}^n} = u, \quad u \in C^m(\mathbb{H}^n)$$

4. Let $f : \Omega \to \mathbb{R}$ be measurable. Define

$$\mu_f(t) := \left| \{ x \in \Omega : |f(x)| > t \} \right|$$

Let $p > 0$ and assume $f \in L_p(\Omega)$. Prove that

$$\mu_f(t) \leq t^{-p} \| f \|_p^p$$

and that

$$\| f \|_p^p = p \int_0^\infty t^{p-1} \mu_f(t) \, dt.$$  

5. Let $1 \leq q < r < \infty$ and $T : L_q(\Omega) \cap L_r(\Omega) \to L_q(\Omega) \cap L_r(\Omega)$ be a linear operator such that

$$\mu_{Tf}(t) \leq (T_1 \| f \|_q / t)^q \quad \text{and} \quad \mu_{Tf}(t) \leq (T_2 \| f \|_r / t)^r$$

for some constants $T_1$ and $T_2$. Then $T$ can be extended to an operator $T \in L(L_p(\Omega))$ for any $p \in (q, r)$ and

$$\| Tf \|_p \leq c T_1^q T_2^{q-\alpha} \| f \|_p, \quad f \in L_q(\Omega) \cap L_r(\Omega)$$

where $1/p = (1 - \alpha)/r + \alpha/q.$
[Hint: For $s > 0$ use $f = f_X(\|f\| > s) + f_X(\|f\| \leq s) = f_1 + f_2$ to prove that $\mu_T f(t) \leq \mu_{T f_1}(t/2) + \mu_{T f_2}(t/2)$ and then use the previous problem and Fubini’s theorem.]