Assignment 2

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1. Let $C^\infty(-1,1) \ni a \geq a_0 > 0$ and $y \in (-1,1)$. Compute $G(\cdot, y)$ satisfying

$$-\partial_x\big(a(x)\partial_x G(\cdot, y)\big) = \delta_y$$

   and $G(-1,y) = G(1,y) = 0$. Let $f \in L^1(-1,1)$ and show that u defined through

$$u(x) = \int_{-1}^{1} G(x,y)f(y)\,dy\,, \ x \in (-1,1)$$

   solves (in which sense?) the following boundary value problem

$$\begin{cases} -\partial_x\big(a(x)\partial_x u\big) = f \\ u(-1) = u(1) = 0 \end{cases}$$

2. Let $\omega \in \mathbb{R}$ and find the solution of

$$(\partial_x^2 + \omega^2)G = \delta \text{ in } \mathcal{D}'(\mathbb{R})$$

   [Hint: Let $G = Hf$ for $f \in C^\infty(\mathbb{R})$ and the Heaviside function H]

3. Let $T \in \mathcal{E}'(\mathbb{R}^n)$ with $\mathrm{supp}(T) = K$ be of finite order $m \in \mathbb{N}$. Show that

$$\langle T, \phi \rangle = 0 \text{ for any } \phi \in \mathcal{E}(\mathbb{R}^n) \text{ with } \partial^\alpha \phi\big|_K \equiv 0 \text{ for } |\alpha| \leq m$$

4. Let E be a given vector space. A function $p : E \to [0,\infty)$ is called *seminorm* if

$$p(x+y) \leq p(x) + p(y) \text{ and } p(\alpha x) = |\alpha|\,p(x)$$

   for $x, y \in E$ and $\alpha \in \mathbb{K}$. A family of seminorms $\{p_\lambda : \lambda \in \Lambda\}$ is called *separating* if

$$p_\lambda(x) = 0 \text{ for each } \lambda \in \Lambda \text{ implies } x = 0\,.$$

   Now, a set $X \subset E$ is called *open* if for each $x \in X$ there are finitely many $\lambda_j \in \Lambda$ and $\epsilon_j > 0$ such that

$$x + \cap_{j=1,\ldots,m}\, p_{\lambda_j}^{-1}\big((0,\epsilon_j)\big) \subset X\,.$$

   Then the collection of all open sets defines a topology on E and E endowed with this topology is called *locally convex space*. A map $u : E \to E$ is then continuous iff preimages of open sets are open. Express this in terms of the seminorms p_λ. Prove that the space $\mathcal{E}(\Omega)$ is a locally convex space with seminorms

$$\big\{p_{K,m} \ : \ K \subset \Omega \text{ compact and } m \in \mathbb{N}\big\}$$

   defined through $p_{K,m}(\varphi) = \sup_{x \in K, |\alpha| \leq m} \big|D^\alpha \varphi(x)\big|\,.$

5. Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi = 1$ on $\mathbb{B}(0,1)$ and define ψ_k through

$$\psi_k(x) = \psi\left(\frac{x}{k}\right), \ x \in \mathbb{R}^n \,.$$

Show that $\psi_k \to \mathbf{1} \ (k \to \infty)$ in $\mathcal{E}(\mathbb{R}^n)$. Then prove that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{E}(\mathbb{R}^n)$ as well as $\mathcal{E}(\mathbb{R}^n)'$ in $\mathcal{D}(\mathbb{R}^n)'$.

The Homework is due by October 19 2001