1. Compute the Fourier transform with respect to the variable \( x \in \mathbb{R} \) of the following functions:
   (i) \( u_y(x) = e^{-y|x|}, y > 0 \).
   (ii) \( u(x) = e^{-|x|^2/2} \).
   (iii) \( u(x) = \frac{y^2-x^2}{(y^2+x^2)^2}, y > 0 \).

2. Show that \( G \) defined through
   \[
   G(x,y) = \frac{1}{\pi} \frac{y}{x^2+y^2}
   \]
   for \( x \in \mathbb{R} \) and \( y > 0 \) is harmonic, that is, \( \Delta G = 0 \), and conclude that
   \[
   u_g(x,y) := \int_{-\infty}^{\infty} G(x-\tilde{x},y)g(\tilde{x}) \, d\tilde{x}, \quad (x,y) \in \mathbb{R} \times (0,\infty)
   \]
   represents a solution of
   \[
   \begin{cases}
   \Delta u = 0 & \text{in } \mathbb{R} \times (0,\infty) \\
   u = g & \text{on } \mathbb{R} \times \{0\}
   \end{cases}
   \]
   for \( g \in L_1(\mathbb{R}) \). What is \( \lim_{y \to \infty} u_g(\cdot,y) \)?

3. Let \( f \in S(\mathbb{R}^n) \) with \( \text{supp}(f) \subset B(0,R) \) for \( R > 0 \). Show that \( \hat{f} \) is holomorphic in \( \xi \in \mathbb{C}^n \) and satisfies
   \[
   |\hat{f}(\xi + i\eta)| \leq c_n \frac{1}{(1+|\xi|^2)^{N/2}} e^{R|\eta|}, \quad (\xi,\eta) \in \mathbb{R}^n.
   \]
   Define \( S^1 = \{ x \in \mathbb{R}^2 | |x| = 1 \} \) and \( \mathbb{T}^n := (S^1)^n \) and assume \( u \in L_1(\mathbb{T}^n) \), that is, assume that \( u \) is an integrable \( 2\pi \) periodic function of all its variables, which can therefore be identified with a function defined on the \( n \)-dimensional torus \( \mathbb{T}^n \). Then its Fourier transform is given by
   \[
   (\mathcal{F}(u))(k) := \hat{u}(k) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ik \cdot \theta} u(\theta) \, d\theta.
   \]
   In analogy to the Fourier transform introduced in class one has that
   \[
   \mathcal{F} \in L(\mathcal{L}(L_1(\mathbb{T}^n), l_\infty(\mathbb{Z}^n))),
   \]
   where \( l_\infty(\mathbb{Z}^n) \) is the Banach space of bounded sequences.
4. Let $n = 1$ and prove that
\[
(S_n u)(\theta) := \sum_{k=-n}^{n} \hat{u}(k)e^{ik\theta} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta - \varphi) D_n(\varphi) \, d\varphi
\]
for $D_n(\theta) = \frac{\sin\left(\frac{(n+1/2)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$.

5. Show that
\[
(S_n u)(0) \to 0 \quad (n \to \infty)
\]
if $g \in L_1(-\pi, \pi)$ for $g(\theta) = f(\theta)/\theta$ and deduce that
\[
(S_n u)(\theta_0) \to u(\theta_0) \quad (n \to \infty)
\]
if $u$ is Lipschitz continuous at $\theta = \theta_0$.

The Homework is due by November 2 2001