Assignment 3

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1. Compute the Fourier transorm with respect to the variable $x \in \mathbb{R}$ of the following functions:
   (i) $u_y(x) = e^{-y|x|}$, $y > 0$.
   (ii) $u(x) = e^{-\frac{|x|^2}{2}}$.
   (iii) $u(x) = \frac{y^2 - x^2}{(y^2+x^2)^2}$, $y > 0$.

2. Show that G defined through $G(x, y) = \frac{1}{\pi} \frac{y}{x^2+y^2}$ for $x \in \mathbb{R}$ and $y > 0$ is harmonic, that is, $\triangle G = 0$, and conclude that
$$u_g(x, y) := \int_{-\infty}^{\infty} G(x - \tilde{x}, y) g(\tilde{x})\, d\tilde{x}\,,\ (x, y) \in \mathbb{R} \times (0, \infty)$$
represents a solution of
$$\begin{cases} \triangle u & = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u & = g \text{ on } \mathbb{R} \times \{0\} \end{cases}$$
for $g \in \mathrm{L}_1(\mathbb{R})$. What is $\lim_{y\to\infty} u_g(\cdot, y)$?

3. Let $f \in \mathcal{S}(\mathbb{R}^n)$ with $\mathrm{supp}(f) \subset \mathbb{B}(0, R)$ for $R > 0$. Show that $\hat{f}$ is holomorphic in $\xi \in \mathbb{C}^n$ and satisfies
$$|\hat{f}(\xi + i\eta)| \leq c_N \frac{1}{(1 + |\xi|^2)^{N/2}} e^{R|\eta|}\,,\ (\xi, \eta) \in \mathbb{R}^n\,.$$

Define $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \,|\, |x| = 1\}$ and $\mathbb{T}^n := (\mathbb{S}^1)^n$ and assume $u \in \mathrm{L}_1(\mathbb{T}^n)$, that is, assume that u is an integrable 2π periodic function of all its variables, which can therefore be identified with a function defined on the n-dimensional torus $\mathbb{T}^n$. Then its Fourier transform is given by
$$\big(\mathcal{F}(u)\big)(k) := \hat{u}(k) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ik\cdot\theta} u(\theta)\, d\theta\,.$$
In analogy to the Fourier transform introduced in class one has that
$$\mathcal{F} \in \mathcal{L}\big(\mathrm{L}_1(\mathbb{T}^n), l_\infty(\mathbb{Z}^n)\big)\,,$$
where $l_\infty(\mathbb{Z}^n)$ is the Banach space of bounded sequences.

4. Let $n = 1$ and prove that
$$\big(S_n u\big)(\theta) := \sum_{k=-n}^{n} \hat{u}(k) e^{ik\cdot\theta} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta - \varphi) D_n(\varphi)\, d\varphi$$
for $D_n(\theta) = \frac{\sin\big((n+1/2)\theta\big)}{\sin(\frac{\theta}{2})}$.

5. Show that
$$\big(S_n u\big)(0) \to 0 \; (n \to \infty)$$
if $g \in \mathrm{L}_1(-\pi, \pi)$ for $g(\theta) = f(\theta)/\theta$ and deduce that
$$\big(S_n u\big)(\theta_0) \to u(\theta_0) \; (n \to \infty)$$
if u is Lipscitz continuous at $\theta = \theta_0$.

The Homework is due by November 2 2001