Assignment 5

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1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and prove *Poincaré's Inequality*
$$\|u\|_{L_2(\Omega)} \leq c \|\nabla u\|_{L_2(\Omega)}\,,\ u \in \overset{\circ}{H}{}^1(\Omega)\,.$$
What does the constant c depend on? Is the boundedness assumption really necessary?

2. Let $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty)$ and define
$$W_p^m(\Omega) = \big\{ u \in \mathcal{D}'(\Omega) \,\big|\, \partial^\alpha u \in L_p(\Omega)\,,\ |\alpha| \leq m \big\}\,.$$
Show that $W_p^m(\Omega)$ is a Banach space if endowed with the norm $\|\cdot\|_{m,p}$ defined by
$$\|u\|_{m,p} = \big(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|^p_{L_p(\Omega)} \big)^{1/p}\,,\ u \in W_p^m(\Omega)\,.$$
Prove that $W_p^1(0,1) \hookrightarrow BUC^{1-1/p}([0,1])$.
[Hint: Use the fact that $C^1([0,1])$ is dense in $W_p^1(0,1)$]

3. Prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$.

4. Let $\Omega = \mathbb{B}(0, 1/2)$ and the function u be defined through
$$u(x,y) = \log\big(\log(\frac{2}{\sqrt{x^2+y^2}})\big)\,,\ (x,y) \in \Omega\,.$$
Then u is obviously not continuous in $(x,y) = (0,0)$. Prove that, however, $u \in H^1(\Omega)$. Let now
$$u(x,y) = xy\,\big[\log\big|\log|(x,y)|\big| - \log\log 2\big]\,,\ (x,y) \in \Omega.$$
Then
$$u \in C^1(\bar\Omega) \text{ and } \partial_j^2 u \in C(\bar\Omega)\,,\ j = 1,2$$
but $u \notin C^2(\bar\Omega)$, that is, u is a solution of the Dirichlet problem in Ω for a continuous datum but is not twice continuously differentiable.

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and prove that the norm $|||\cdot|||_{L_2(\Omega)} = \|\triangle\,\cdot\|_{L_2(\Omega)}$ is an equivalent norm on $\overset{\circ}{H}{}^2(\Omega)$. Find the natural weak formulation of the bvp
$$\triangle^2 u = f \in L_2(\Omega)\,,\ u = 0\,,\ \partial_\nu u = 0 \text{ on } \partial\Omega$$
and prove that it has a unique solution $u \in \overset{\circ}{H}{}^2(\Omega)$.

The Homework is due by November 30 2001