1. Consider the following porous medium equation:

\[ u_t - \Delta (u^γ) = 0 \text{ in } \mathbb{R}^n \times (0, \infty). \]

(i) Show that there exists a solution of the form \( u(t, x) = v(t)w(x) \).

(ii) Find a solution of the form \( u(t, x) = \frac{1}{t^\alpha} v \left( \frac{x}{t^\beta} \right) \).

What are the main features of these solutions?

2. Find a transformation \( w = \Phi(u) \) which reduces the nonlinear equation

\[
\begin{cases}
  u_t - a\Delta u + b|\nabla u|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
  u = g & \text{in } \mathbb{R}^n \times \{0\}
\end{cases}
\]

to a linear one and use the latter to find a solution of the former.

3. Use the solution to the previous problem to find a solution for the viscous Burger’s equation

\[
\begin{cases}
  u_t - au_{xx} + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\
  u = g & \text{in } \mathbb{R} \times \{0\}
\end{cases}
\]

[Hint: Derive an equation for \( w(t, x) = \int_{-\infty}^x u(t, y) \, dy \).]

4. Consider Euler’s equation

\[
\begin{cases}
  u_t - u \cdot Du = -\nabla p & \text{in } \mathbb{R}^3 \times (0, \infty), \\
  \text{div} \, u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\
  u = g & \text{in } \mathbb{R}^3 \times \{0\}.
\end{cases}
\]

Assume that there exists \( v \) such that \( u = \nabla v \) and derive a simpler equation for \( v \). Once \( v \) is obtained, how can \( p \) be derived?

5. Consider the minimal surface equation

\[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \text{ in } \mathbb{R}^2 \]

Let \( |D^2u| \neq 0 \) at \( x_0 \in \mathbb{R}^2 \) and derive the equation satisfied by

\[ v(p) := x(p) \cdot p - u(x(p)) \]

in a neighborhood of \( x_0 \), where \( x(p) \) is the unique solution of

\[ p = \nabla u(x). \]

Homework due Friday, April 29 2005