Let $\Omega \subset \mathbb{R}^n$ be open. A map $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory function whenever
(i) $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}$.
(ii) $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \Omega$.

1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $p, q \geq 1$. Assume that
\[|f(x, s)| \leq c|s|^{p/q} + g(x)\]
for some $g \in L_q(\Omega)$. Prove that the Nemytzki operator (substitution operator) $N_f : L_p(\Omega) \to L_q(\Omega)$ defined through
\[(N_f u)(x) := f(x, u(x)) , x \in \Omega\]
is well-defined, continuous and maps bounded sets onto bounded sets.

2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Assume that the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies
\[f \leq \frac{f(x, u) - f(x, v)}{u - v} \leq \overline{f} \text{ and } f(\cdot, 0) \in L_2(\Omega)\]
with $\sigma(-\Delta_D) \cap [\overline{f}, \overline{f}] = \emptyset$. Show that
\[\begin{cases} 
\Delta u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega 
\end{cases}\]
possesses a unique weak solution $u \in H^1(\Omega)$.

3. Let $H$ be a Hilbert space. Prove that
\[x_n \to x , y_n \rightharpoonup y (n \to \infty) \Rightarrow (x_n|y_n) \to (x|y) (n \to \infty) .\]

4. Let $E$ be a normed vector space and $A \in \mathcal{L}(E)$ a compact operator. Show that
\[\dim(ker[(\lambda - A)^n]) < \infty \text{ for any } \lambda \neq 0 \text{ and } n = 1, 2, \ldots\]
[Use the fact that $\mathcal{B}_E(0, 1)$ is compact iff $\dim(E) < \infty$ and that eigenvectors to different eigenvalues are linearly independent.]

5. Let $K \in \mathcal{L}(H)$ be a compact operator. Show that it maps weakly convergent sequences to convergent sequences.

Homework due on Friday, May 27 2005