Assignment 2

1. Let $C^\infty[-1,1] \ni a \geq a_0 > 0$ and $y \in [-1,1]$. Compute $G(\cdot, y)$ satisfying
   \[-\partial_x (a(x) \partial_x G(\cdot, y)) = \delta_y\]
   and $G(-1, y) = G(1, y) = 0$. Let $f \in L^1(-1,1)$ and show that $u$ defined through
   \[u(x) = \int_{-1}^{1} G(x, y) f(y) \, dy, \quad x \in (-1,1)\]
   solves (in which sense?) the following boundary value problem
   \[
   \begin{cases}
   -\partial_x (a(x) \partial_x u) = f(x), & x \in (-1,1), \\
   u(-1) = u(1) = 0.
   \end{cases}
   \]

2. Let $\omega \in \mathbb{R}$ and find the solution of
   \[(\partial_x^2 + \omega^2) G = \delta \text{ in } D'(\mathbb{R})\]
   [Hint: Let $G = Hf$ for $f \in C^\infty(\mathbb{R})$ and the Heaviside function $H$]

3. Let $T \in \mathcal{E}'(\mathbb{R}^n)$ with $\text{supp}(T) = K$ be of finite order $m \in \mathbb{N}$. Show that
   \[\langle T, \phi \rangle = 0 \text{ for any } \phi \in \mathcal{E}(\mathbb{R}^n) \text{ with } \partial^\alpha \phi|_K \equiv 0 \text{ for } |\alpha| \leq m\]

4. Let $E$ be a given vector space. A function $p : E \to [0, \infty)$ is called seminorm if
   \[p(x + y) \leq p(x) + p(y) \text{ and } p(\alpha x) = |\alpha| p(x)\]
   for $x, y \in E$ and $\alpha \in \mathbb{K}$. A family of seminorms $\{p_\lambda : \lambda \in \Lambda\}$ is called separating if
   \[p_\lambda(x) = 0 \text{ for each } \lambda \in \Lambda \text{ implies } x = 0.\]
   Now, a set $X \subset E$ is called open if for each $x \in X$ there are finitely many $\lambda_j \in \Lambda$ and $\epsilon_j > 0$ such that
   \[x + \cap_{j=1,\ldots,m} p_{\lambda_j}^{-1}((0, \epsilon_j)) \subset X.\]
   Show that the collection of all so-defined open sets is a topology on $E$. $E$ endowed with this topology is called locally convex space. A linear map
   \[u : E \to E\]
   is then continuous if preimages of open sets are open. Find a characterization of its continuity by means of inequalities for the seminorms
Prove that the space $\mathcal{E}(\Omega)$ defined in class is a locally convex space with seminorms
$$\{ p_{K,m} : K = \bar{K} \subset \subset \Omega, m \in \mathbb{N} \}$$
defined through $p_{K,m}(\varphi) = \sup_{x \in K, |\alpha| \leq m} |D^\alpha \varphi(x)|$.

5. Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi = 1$ on $B(0,1)$ and define $\psi_k$ through
$$\psi_k(x) = \psi\left(\frac{x}{k}\right), \quad x \in \mathbb{R}^n.$$  
Show that $\psi_k \to 1 \ (k \to \infty)$ in $\mathcal{E}(\mathbb{R}^n)$. Then prove that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{E}(\mathbb{R}^n)$ as well as $\mathcal{E}(\mathbb{R}^n)'$ in $\mathcal{D}(\mathbb{R}^n)'$.