Assignment 4

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1. Compute a fundamental solution G_2 for the wave operator $\partial_t^2 - \triangle$ on $\mathbb{R} \times \mathbb{R}^2$. [Hint: Let G_3 be the fundamental solution for the wave operator on $\mathbb{R} \times \mathbb{R}^3$ introduced in class and show that
$$\langle G_3(t, x, x_3), \varphi(t, x)\, \mathbf{1}(x_3) \rangle$$
can be made sense of for $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$. Define G_2 through
$$\langle G_2, \varphi \rangle = \langle G_3, \varphi\, \mathbf{1}(x_3) \rangle = \frac{1}{4\pi} \int_0^\infty \frac{1}{t} \int_{\mathbb{S}_t^2} \varphi(t, x)\, d\sigma_{\mathbb{S}_t^2}(x) dt$$
and use the fact that φ is independent of x_3 to simplify the integral.]

2. A fundamental solution $G(\cdot, y)$ for the Laplacian $-\triangle$ on a domain $\Omega \subset \mathbb{R}^n$ with pole at $x = y$ and satisfying $G(\cdot, y)\big|_{\partial\Omega} = 0$ is called *Green's function* for the Dirichlet problem in Ω (considered as a function of $(x, y) \in \Omega \times \Omega$). Compute a Green's function for the Dirichlet problem in the half-space $\mathbb{H}^n = \mathbb{R}^{n-1} \times (0, \infty)$, $n \geq 2$.

3. Let G be the fundamental solution for the heat operator $\partial_t - \triangle$ on $\mathbb{R} \times \mathbb{R}^n$ introduced in class. Show that
$$G(t, \cdot) \to \delta\,,\ t \to 0+\,,\ \text{in } \mathcal{S}'(\mathbb{R}^n)$$
and find a representation for the solution of the Cauchy problem
$$\partial_t u - \triangle u = f \in \mathrm{L}_{1,loc}\big([0, \infty), \mathcal{S}'\big)\,,\ u(0, \cdot) = u_0 \in \mathcal{S}'$$
on $(0, \infty) \times \mathbb{R}^n$ in terms of G.

4. Compute the general solution to the following initial value problem for the wave equation
$$\partial_t^2 u - \partial_x^2 u = f(t, x)\,,\ u(0, \cdot) = u_0\,,\ \partial_t u(0, \cdot) = u_1\,,$$
in $(0, \infty) \times \mathbb{R}$ in terms of a fundamental solution for the wave operator. What is the regularity of the solution if
$$(f, u_0, u_1) \in \mathrm{C}\big([0, \infty), \mathcal{S}\big) \times \mathcal{S} \times \mathcal{S}\,?$$

5. Let $u : \Omega \to \mathbb{R}$ be a harmonic function on the domain Ω. Prove *Gauss's law of the arithmetic mean*
$$u(x) = \frac{1}{\omega_n\, r^{n-1}} \int_{|x-y|=r} u(y)\, d\sigma_r(y)$$
valid for all x and r such that $\mathbb{B}(x, r) \subset \Omega$.
[Hint: Use Green's formula with u and a Green's function for the Dirichlet problem on $\mathbb{B}(x, r)$.]

Homework due by Friday, November 19