Assignment 8

1. Let $T$ be a $c_0$-semigroup on a Banach space $E$. Prove that there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that
   \[ \|T(t)\|_{\mathcal{L}(E)} \leq M e^{\omega t}. \]

2. For a $c_0$-semigroup $T$ on a Banach space $E$ define \( \xi(t, x) = T(t)x, \ t \in [0, \infty), \ x \in E \) and prove that the following are equivalent:
   (i) $\xi(\cdot, x)$ is differentiable.
   (ii) $\xi(\cdot, x)$ is right differentiable.

3. Show that the translation semigroup $T$ on $\text{BUC}(\mathbb{R})$ defined through
   \[ T(t)f(\cdot) = f(\cdot - t), \ f \in \text{BUC}(\mathbb{R}) \]
   is strongly continuous and compute its generator.

4. Let $A \in \mathcal{G}(E), \ x \in E$ and $f \in C^1([0, \infty) \times E, E)$ and prove that
   \[
   \begin{cases}
   \dot{u} + Au = f(t, u), \ t > 0 \\
   u(0) = x
   \end{cases}
   \]
   has a unique local mild solution $u(\cdot, x) \in C([0, t^+(x)), E)$ for some $t^+(x) > 0$.
   [Hint: Use Banach Fixed-Point Theorem combined with the Variation-of-Constants-Formula]

5. Let $A \in \mathbb{C}^{n \times n}$ and show that
   \[ e^{-tA} = \frac{1}{2\pi i} \int_{\partial B(0, R)} e^{\lambda t}(\lambda + A)^{-1} \ d\lambda, \]
   where $R > 0$ is such that $\sigma(-A) \subset B(0, R)$ and the integration is counterclockwise.
   [Hint: Show that both sides of the equation satisfy the same ODE and prove that they have the same initial value by reducing the problem to the case where $A$ is a Jordan block]

The Homework is due February 18 2005