Assignment 10

1. Let $0 \neq \alpha \in \mathbb{R}$ and consider the initial value problem for Euler's equation

$$\sum_{k=1}^{n} x_k \partial_{x_k} u = \alpha u, \ u(x_1, \dots, x_{n-1}, 1) = g(x_1, \dots, x_{n-1}).$$

Show that its solution satisfies the functional equation

 $u(\lambda x) = \lambda^{\alpha} u(x), \ x \neq 0, \ \lambda > 0.$

What is the behavior of the solution at x = 0?

2. Consider the quasilinear initial value problem

$$\begin{cases} u_t + u \, u_x = 0, & (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, x) = g(x), & x \in \mathbb{R} \end{cases}$$

Solve the equation and analyze the possible onset of singularities. What conditions on g would prevent the solution from developing singularities?

3. Let $u \in C^1(\mathbb{B}(0,1))$ be a solution of

$$a(x,y)u_x + b(x,y)u_y = -u$$

and assume that a(x,y)x + b(x,y)y > 0 for $(x,y) \in \mathbb{S}^1$. Show that $u \equiv 0$ then.

4. Consider the equation

$$\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0$$

Any function u with

$$\int_{\mathbb{R}^2} \left[R(u)\phi_y + S(u)\phi_x \right] d(x,y) = 0 \,, \, \phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$$

is called *weak solution*. Assume that u is continuously differentiable away from some curve parametrized by $(s(y), y), y \in \mathbb{R}$ across which it has a jump discontinuity. Conclude that

$$s'(y) = \frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)}$$

where u^{\pm} indicate the one sided limits of u approaching the curve.

5. Consider the eikonal equation

$$c^2(u_x^2 + u_y^2) = 1$$

Let γ_t be the level line [u(x, y) = t] of a solution u. Show that a point (x, y) moves in a direction perpendicular to γ_t at constant speed c (along a characteristic).

The Homework is due by Wednesday, April 18 2007

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