Assignment 2

1. It can be shown that any distribution can be viewed as some derivative of continuous function. More precisely

\[ \forall u \in \mathcal{D}'(\Omega) \forall K \subset \Omega \exists f \in C(\Omega) \text{ and } \alpha \in \mathbb{N}^n \text{ s.t.} \]

\[ \langle u, \varphi \rangle = (-1)^{|\alpha|} \int_{\Omega} f(x) \partial^\alpha \varphi(x) \, dx \forall \varphi \in \mathcal{D}(\Omega). \]

Let \( \delta \in \mathcal{D}'(\Omega) \) and find a simple representation of it as some derivative of a continuous function.

2. For \( u \in \mathcal{D}'(\mathbb{R}^n) \) and \( v \in \mathbb{R}^n \) define \( \tau^v u \in \mathcal{D}'(\mathbb{R}^n) \) through

\[ \langle \tau^v u, \varphi \rangle = \langle u, \tau^v \varphi \rangle \forall \varphi \in \mathcal{D}(\mathbb{R}^n) \]

where \( \tau^v \varphi = \varphi(\cdot + v) \). Show that \( \tau^v u \) is well-defined and prove that

\[ \frac{\tau_{e_j} u - u}{h} \rightarrow \partial_j \varphi \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ as } h \to 0 \]

for \( j = a, \ldots, n \).

3. Let \( \varphi, \psi \in \mathcal{D}(\Omega) \) and \( \alpha \in \mathbb{N}^n \). Determine the real numbers \( c_{\alpha\beta} \), \( \beta \leq \alpha \) for which the formula

\[ \partial^\alpha (\varphi \psi) = \sum_{\beta \leq \alpha} \partial^{\alpha - \beta} \varphi \partial^\beta \psi \]

holds. By \( \beta \leq \alpha \) it is meant that \( \beta_j \leq \alpha_j \) for \( j = a, \ldots, n \).

4. Show that \( u \in \mathcal{D}'((0, \infty)) \) given through

\[ \langle u, \varphi \rangle := \sum_{m \in \mathbb{N}} \left( \partial^m \varphi \right) \left( \frac{1}{m} \right), \varphi \in \mathcal{D}((0, \infty)) \]

is well-defined. Show that \( u \) has infinite order and that it cannot be extended to a distribution of the real line.

5. Let \( \Omega \subset \mathbb{R}^n \) be an open and bounded domain with smooth boundary \( \partial \Omega \) and denote by \( \chi_\Omega \) its characteristic function. Compute \( \nabla \chi_\Omega \) in the sense of distributions and determine its support.

Homework due by Wednesday, October 18 2006