Assignment 4

1. Compute the Fourier transform with respect to the variable $x$ of the following functions:
   (i) $u_y(x) = e^{-y|x|}$, $y > 0$, $x \in \mathbb{R}$.
   (ii) $u(x) = e^{-\frac{|x|^2}{2}}$, $x \in \mathbb{R}^n$.
   (iii) $u(x) = \frac{y^2 - x^2}{(y^2 + x^2)^2}$, $y > 0$, $x \in \mathbb{R}$.

2. Show that $G$ defined through $G(x, y) = \frac{1}{\pi} \frac{y - x^2}{y^2 + x^2}$ for $x \in \mathbb{R}$ and $y > 0$ is harmonic, that is, $\Delta G = 0$, and conclude that
   \[ u_g(x, y) := \int_{-\infty}^{\infty} G(x - \tilde{x}, y) g(\tilde{x}) \, d\tilde{x} , \quad (x, y) \in \mathbb{R} \times (0, \infty) \]
   represents a solution of
   \[
   \begin{cases}
   \Delta u &= 0 \text{ in } \mathbb{R} \times (0, \infty) \\
   u &= g \text{ on } \mathbb{R} \times \{0\}
   \end{cases}
   \]
   for $g \in L_1(\mathbb{R})$. What is $\lim_{y \to \infty} u_g(\cdot, y)$?

3. Let $f \in S(\mathbb{R}^n)$ with supp$(f) \subset B(0, R)$ for $0 < R < \infty$. Show that its Fourier transform $\hat{f}$ is holomorphic and satisfies
   \[ |\hat{f}(\xi + i\eta)| \leq c_N \frac{1}{(1 + |\xi|^2)^{N/2}} e^{R|\eta|}, \quad (\xi, \eta) \in \mathbb{R}^{2n}. \]

4. Assume $\varphi \in S(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ and let
   \[ T : \mathbb{R} \to S(\mathbb{R}^n), \quad t \to \varphi(\cdot - ta). \]
   Prove that $T \in C^1(\mathbb{R}, S(\mathbb{R}^n))$ and compute
   \[ \hat{T}(0) \in \mathcal{L}(\mathbb{R}, S(\mathbb{R}^n)) \cong S(\mathbb{R}^n). \]

5. Let $u_0 \in S(\mathbb{R}^n)$ and consider the homogeneous heat equation
   \[
   \begin{cases}
   u_t - \Delta u &= 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n \\
   u(0) &= u_0, \quad \text{in } \mathbb{R}^n
   \end{cases}
   \]
   Prove that it has a unique solution
   \[ u \in C^\infty([0, \infty), S(\mathbb{R}^n)) \]
   and derive a representation formula for it.

Homework due by Wednesday, November 8 2006