

## Assignment 7

---

1. For  $\alpha \in (0, 1)$ ,  $0 \leq s_0 < s_1$  and  $s = (1 - \alpha)s_0 + \alpha s_1$  prove the interpolation inequality

$$\|u\|_{\mathbb{H}^s} \leq c \|u\|_{\mathbb{H}^{s_0}}^{1-\alpha} \|u\|_{\mathbb{H}^{s_1}}^\alpha, \quad u \in \mathbb{H}^{s_1}$$

first for  $\mathbb{R}^n$  and then for a bounded domain with smooth boundary.

2. Let Banach spaces  $E_j$ ,  $j = 0, 1, 2$ , be given with

$$E_2 \hookrightarrow E_1 \hookrightarrow E_0.$$

Show that, given  $\varepsilon > 0$ , there is a constant  $c_\varepsilon > 0$  such that

$$\|u\|_{E_1} \leq \varepsilon \|u\|_{E_2} + c_\varepsilon \|u\|_{E_0}, \quad u \in E_2.$$

3. Show that the trace operator  $\gamma_{\mathbb{H}^n}$  satisfies

$$\gamma_{\mathbb{H}^n}(\mathbb{H}^2(\mathbb{R}^n)) = \mathbb{H}^{3/2}(\partial\mathbb{H}^n).$$

4. Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary,  $\alpha > 0$  and  $f \in L_2(\Omega)$ . Find the boundary value problem for which

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} uv \, dx + \alpha \int_{\partial\Omega} uv \, d\sigma = \int_{\omega} fv \, dx \quad \forall v \in \mathbb{H}^1(\Omega)$$

is the appropriate weak formulation and show that it possesses a unique solution.

5. Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary and find the natural weak formulation of the bvp

$$\Delta^2 u = f \in L_2(\Omega), \quad u = 0, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega$$

and prove that it has a unique solution.